

# HAMMERSLEY'S HARNESS PROCESS: INVARIANT DISTRIBUTIONS AND HEIGHT FLUCTUATIONS

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**ABSTRACT.** We study the invariant distributions of Hammersley's serial harness process in all dimensions and height fluctuations in one dimension. Subject to mild moment assumptions there is essentially one unique invariant distribution, and all other invariant distributions are obtained by adding harmonic functions of the averaging kernel. We identify one Gaussian case where the invariant distribution is i.i.d. Height fluctuations in one dimension obey the stochastic heat equation with additive noise (Edwards-Wilkinson universality). We prove this for correlated initial data subject to polynomial decay of strong mixing coefficients, including process-level tightness in the Skorohod space of space-time trajectories.

## 1. INTRODUCTION

Thinking about the crystalline structure of metals around 1956 led J. M. Hammersley to formulate the *serial harness*. This process  $h_t(x)$  evolves on the lattice  $\mathbb{Z}^d$  via the equation

$$(1.1) \quad h_{t+1}(x) = \sum_y p(y-x)h_t(y) + \xi_{t+1}(x)$$

where  $p$  is a symmetric random walk kernel on  $\mathbb{Z}^d$  and  $\{\xi_t(x)\}$  are i.i.d. random variables with mean zero and finite variance. These ideas were recorded later in article [15] that contains, among other things, calculations that relate the fluctuations of the process to the behavior of the random walk kernel.

Toom [35] studied the convergence of  $h_t(x)$  as  $t \rightarrow \infty$ , as a function of the tail of the noise  $\xi_t(x)$ . [35] also contains several references to the physics literature.

A natural continuous-time version of the process applies the local update (1.1) at the epochs of Poisson processes attached to lattice points  $x$ . The ergodic properties of the continuous-time process were investigated by Hsiao [17, 18], first for the Gaussian case and then more generally. In particular, [18] recorded the order of magnitude of the fluctuations of  $h_t(x)$  from the identically 0 initial condition:  $t^{1/4}$  in  $d = 1$ ,  $\sqrt{\log t}$  in  $d = 2$ , and bounded in  $d \geq 3$ . In  $d \geq 3$ , [18] showed (i) uniqueness of an invariant distribution that is invariant under spatial shifts and has given finite mean and variance, and (ii) convergence to such equilibrium from initial distributions that are invariant and ergodic under spatial shifts and possess a finite second moment.

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*Date:* April 28, 2015.

*Key words and phrases.* Harness, Gaussian process, Edwards-Wilkinson universality class, random walk, fractional Brownian motion, fluctuations, interface, process tightness, strong mixing coefficients, linear process, harmonic crystal, stochastic heat equation.

In the same continuous-time setting, Ferrari and Niederhauser [13] introduced a dual representation of the harness process in terms of backward random walks. Through martingale techniques and the random walk representation, [13] proved convergence of the process from flat initial profiles both on  $\mathbb{Z}^d$  and on subsets of  $\mathbb{Z}^d$  and obtained some rates of convergence. In  $d \in \{1, 2\}$ , instead of  $h_t$  itself, [13] considered  $h_t - h_t(0)$  (the process as seen from the height at the origin) or the pinned process with boundary condition  $h_t(0) = 0$ . In particular, [13] identified the invariant distributions of the harness with Gaussian noise in  $d \geq 3$  as Gaussian Gibbs fields (harmonic crystals) studied by [6].

Our paper works with the discrete time process (1.1), with a general finitely supported random walk kernel  $p$ . We establish that in one space dimension the height fluctuations of the process obey Edwards-Wilkinson universality. This means that on the space and time scale  $n$ , fluctuations of the height are of order  $n^{1/4}$ , spatial correlations occur on the scale  $n^{1/2}$ , and limit distributions are Gaussian. In particular, height fluctuations of the stationary process converge to fractional Brownian motion with Hurst parameter  $1/4$ . We do not address limits in higher dimensions.

We prove the height fluctuations for spatially stationary, correlated initial data, subject to polynomial decay of strong mixing coefficients. Our proofs use the discrete-time counterpart of the dual representation of Ferrari and Niederhauser [13] and employ a CLT for linear processes due to Peligrad and Utev [26]. We obtain also process-level tightness in the Skorohod space of space-time trajectories.

As preparation for the fluctuation results we study the invariant distributions of the process. In this part there is no advantage in restricting to one dimension so we do it in general. The difference with past work is that we consider the increment process, as is necessary for invariant distributions in  $d \in \{1, 2\}$ . In any case, the increment process is the natural object to study because total increment is conserved. (In one dimension, if height  $h(x)$  jumps, the changes to the left and right increments  $h(x) - h(x-1)$  and  $h(x+1) - h(x)$  cancel each other.) For a given kernel  $p$  and noise distribution  $\xi$ , there is essentially one invariant distribution and its transformations by adding harmonic functions. We prove this uniqueness under a condition on the growth of the first moment of the increments on the lattice  $\mathbb{Z}^d$ .

To summarize, these are the contributions of this paper.

(i) For invariant distributions we cover low dimensions  $d = \{1, 2\}$ . Our convergence and uniqueness results require weaker moment assumptions than earlier results and we do not assume spatial shift invariance for uniqueness.

(ii) We prove that the harness process obeys EW universality. For EW class height fluctuations in general, past work is for product form initial distributions, while our paper covers some correlated initial data. In particular, we see that fractional Brownian motion with Hurst parameter  $1/4$  arises from a stationary process also when the invariant distribution is not of product form. We also obtain process-level tightness in space and time, which was done earlier only for independent walks.

Past proofs of one-dimensional EW class fluctuations for an interface or a particle current covered independent particles [21, 32, 33], independent particles in a static random environment [27] and in a dynamic random environment [20], the random average process [2, 12], and a recent continuum example from Howitt-Warren flows [37]. In the context of

colliding particles where the position of a tagged particle is a function of current, related work goes back to Harris [16] and later in [9]. Fluctuations of order  $t^{1/4}$  and fractional Brownian motion limits have also been identified in the simple symmetric exclusion process since the classic work of Arratia [1] and later in [8, 19, 25, 30].

The EW class should be contrasted with the KPZ (Kardar-Parisi-Zhang) class where the exponents are  $1/3$  for height or current fluctuations and  $2/3$  for spatial correlations. Recent reviews of KPZ appear in [4, 7]. Roughly speaking, EW fluctuations are expected when the macroscopic velocity of the interface is a linear function of the slope (as in Theorem 2.1 below), while KPZ fluctuations are expected when this connection is nonlinear.

**Organization of the paper.** Section 2 describes the model and the results on the invariant distributions, and Section 3 contains the height fluctuations in  $d = 1$ . The remainder of the paper covers the proofs: Section 4 for invariant distributions, Section 5 for finite-dimensional convergence of height fluctuations, and Section 6 for process-level tightness of height fluctuations.

**Notation and conventions.** We collect here some items for quick reference.  $\mathbb{Z}_+ = \{0, 1, 2, 3, \dots\}$ ,  $\mathbb{N} = \{1, 2, 3, \dots\}$ ,  $[n] = \{1, 2, \dots, n\}$ . The  $d$ -dimensional integer lattice is  $\mathbb{Z}^d = \{x = (x_1, \dots, x_d) : \text{each } x_i \in \mathbb{Z}\}$ .  $|x|$  is the absolute value of  $x \in \mathbb{R}$  or the Euclidean norm of  $x \in \mathbb{R}^d$ . The imaginary unit is  $\iota = \sqrt{-1}$ .  $C$  is a finite positive constant whose value may change from line to line.

$X \sim \mu$  means that random variable  $X$  has distribution  $\mu$ :  $P(X \in A) = \mu(A)$  for Borel sets  $A$  on the state space of  $X$ .  $\mathcal{M}_1$  is the space of Borel probability measures on  $(\mathbb{R}^d)^{\mathbb{Z}^d}$ .  $\mathcal{I}$  is the subspace of measures that are invariant for the increment process defined below by equation (2.14). Transition probability  $p^k(x, y)$  can be written  $p_{x,y}^k$  to save space.

$E$  represents generic expectation, and  $E^\mu$  and  $\text{Var}^\mu$  denote expectation and variance under probability measure  $\mu$ . There are three particular expectation operators. (i)  $\mathbb{E}$  is expectation under the probability measure  $\mathbb{P}$  of the i.i.d. variables  $\xi = \{\xi_t(x)\}_{t \in \mathbb{Z}, x \in \mathbb{Z}^d}$  that drive the dynamics. (ii)  $\mathbf{P}$  is the probability measure and  $\mathbf{E}$  the expectation of the harness process. The superscript on  $\mathbf{E}^\eta$  and  $\mathbf{E}^\nu$  identifies the initial increment configuration  $\eta$  or the initial distribution  $\nu$ .  $\mathbb{P}$  is the  $\xi$ -marginal of  $\mathbf{P}$ . (iii) In the proofs of the height fluctuations,  $P$  with expectation  $E$  refers to the random walks  $X_t^i$  of the dual representation of the harness process.

## 2. THE HARNESS PROCESS AND ITS INVARIANT DISTRIBUTIONS

**2.1. The model.** Fix a dimension  $d \in \mathbb{N}$ . The state of the harness process is a real-valued *height function*  $h : \mathbb{Z}^d \rightarrow \mathbb{R}$  that evolves randomly in discrete time. The evolution is determined by a fixed weight vector  $\{p(x)\}_{x \in \mathbb{Z}^d}$  and a collection of i.i.d. “noise” variables  $\{\xi_t(x)\}_{t \in \mathbb{Z}, x \in \mathbb{Z}^d}$ . The weight vector  $\{p(x)\}$  is a finitely supported nondegenerate probability vector on  $\mathbb{Z}^d$  with these properties:

$$(2.1) \quad \begin{aligned} &0 \leq p(x) < 1, \quad \sum_{x \in \mathbb{Z}^d} p(x) = 1, \quad \exists M < \infty \text{ such that } p(x) = 0 \text{ for } |x| > M, \text{ and} \\ &\forall u \in \mathbb{Z}^d, \text{ the smallest additive subgroup of } \mathbb{Z}^d \text{ that contains the translated} \\ &\text{support } \{u + x : p(x) > 0\} \text{ is } \mathbb{Z}^d \text{ itself.} \end{aligned}$$

The last property is *strong aperiodicity* in Spitzer's terminology [34, p. 42]. In  $d = 1$  it is the same as requiring that  $p$  have *span 1* [10, Sect. 3.5].

It is useful to think of  $p(x)$  as a random walk transition probability  $p(x, y) = p(y - x)$ . Multistep transitions are denoted by  $p^0(x, y) = \mathbf{1}_{x=y}$ ,  $p^1(x, y) = p(x, y)$ , and for  $k \geq 2$ ,

$$p^k(x, y) = \sum_{x_1, x_2, \dots, x_{k-1} \in \mathbb{Z}^d} p(x, x_1) p(x_1, x_2) \cdots p(x_{k-1}, y).$$

Let  $v_1 = \sum_{x \in \mathbb{Z}^d} x p(x)$  denote the mean (vector) of  $p$ . In  $d = 1$  the variance is

$$(2.2) \quad \sigma_1^2 = \sum_{x \in \mathbb{Z}} (x - v_1)^2 p(x), \quad \text{assumed } > 0.$$

The state of the height process at time  $t \in \mathbb{Z}$  is denoted by  $h_t = \{h_t(x)\}_{x \in \mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d}$ . Given  $h_t$ , the evolution from time  $t$  to  $t + 1$  is governed by the equation

$$(2.3) \quad h_{t+1}(x) = \sum_{y \in \mathbb{Z}^d} p(x, y) h_t(y) + \xi_{t+1}(x), \quad x \in \mathbb{Z}^d.$$

Since adding a constant to  $\xi_t(x)$  would simply add a constant speed to  $h_t$ , we assume that  $\mathbb{E}[\xi_t(x)] = 0$ . Our results also require (at least) square-integrability. We summarize these assumptions as

$$(2.4) \quad \begin{aligned} &\xi = \{\xi_t(x)\}_{t \in \mathbb{Z}, x \in \mathbb{Z}^d} \text{ are i.i.d. real-valued mean-zero random variables} \\ &\text{with } \sigma_\xi^2 = \mathbb{E}[|\xi_t(x)|^2] < \infty. \end{aligned}$$

The probability measure on the variables  $\xi$  is denoted by  $\mathbb{P}$  and expectation by  $\mathbb{E}$ , while for the harness process we write  $\mathbf{P}$  and  $\mathbf{E}$ .  $\mathbb{P}$  is the  $\xi$ -marginal of  $\mathbf{P}$ .

That the harness process should obey EW universality is indicated by a hydrodynamic limit described by a linear partial differential equation  $\frac{\partial}{\partial t} u - v_1 \cdot \nabla u = 0$ .

**THEOREM 2.1 (Hydrodynamic limit).** *Let  $u_0$  be a continuous function on  $\mathbb{R}^d$  and define  $u(t, x) = u_0(x + tv_1)$ . Let  $\{h^n\}$  be a sequence of harness processes whose initial height functions approximate  $u_0$  locally uniformly in probability:*

$$\lim_{n \rightarrow \infty} \mathbf{P} \left\{ \sup_{x \in \mathbb{R}^d: |x| \leq R} |n^{-1} h_0^n(\lfloor nx \rfloor) - u_0(x)| \geq \varepsilon \right\} = 0 \quad \forall \varepsilon > 0, R < \infty.$$

Then

$$\lim_{n \rightarrow \infty} \mathbf{P} \left\{ |n^{-1} h_{\lfloor nt \rfloor}^n(\lfloor nx \rfloor) - u(t, x)| \geq \varepsilon \right\} = 0 \quad \forall \varepsilon > 0, (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d.$$

We omit the straightforward proof of the theorem (see [38]).

**2.2. Invariant distributions.** We focus on the invariant distributions of the increment process because those exist in all dimensions. Once formulated, the increment process stands on its own and is more general because not every process that obeys the increment evolution comes from a height process. We study the invariant distributions of this more general “increment process”. At the end of the section we make brief contact with the height process.

Given a height process  $h$ , that obeys (2.3), define increment variables by

$$(2.5) \quad \eta_t(x, y) = h_t(y) - h_t(x) \quad \text{for } x, y \in \mathbb{Z}^d.$$

Increment variables satisfy the additivity relation

$$(2.6) \quad \eta_t(x, y) + \eta_t(y, z) = \eta_t(x, z) \quad \text{for } x, y, z \in \mathbb{Z}^d$$

and by (2.3) the Markovian evolution

$$(2.7) \quad \eta_t(x, y) = \sum_{z \in \mathbb{Z}^d} p(0, z) \eta_{t-1}(x + z, y + z) + \xi_t(y) - \xi_t(x).$$

Conversely, if variables  $\eta_t(x, y)$  satisfy additivity (2.6) at  $t = 0$  and (2.7) for  $t \in \mathbb{N}$ , then (2.5) holds for the height process  $h_\bullet$  defined by  $h_0(0) = 0$ ,  $h_0(x) = \eta_0(0, x)$ , and (2.3).

Iterate (2.7) backward in time:

$$(2.8) \quad \begin{aligned} \eta_t(x, y) &= \sum_{z \in \mathbb{Z}^d} p(0, z) \eta_{t-1}(x + z, y + z) + \xi_t(y) - \xi_t(x) \\ &= \cdots = \sum_{z \in \mathbb{Z}^d} p^s(0, z) \eta_{t-s}(x + z, y + z) + \sum_{k=0}^{s-1} \sum_{z \in \mathbb{Z}^d} \xi_{t-k}(z) (p^k(y, z) - p^k(x, z)) \end{aligned}$$

for  $x, y \in \mathbb{Z}^d$  and  $s \in \mathbb{N}$ . Imagine taking  $s \rightarrow \infty$  with zero initial condition “ $\eta_{-\infty} = 0$ ” in the infinite past. This suggests the definition

$$(2.9) \quad \Delta_t(x, y) = \sum_{k=0}^{\infty} \sum_{z \in \mathbb{Z}^d} \xi_{t-k}(z) (p^k(y, z) - p^k(x, z)), \quad x, y \in \mathbb{Z}^d, \quad t \in \mathbb{Z}.$$

The series of independent mean zero variables on the right of (2.9) converges almost surely and in  $L^2(\mathbb{P})$  because the sum of variances converges (Theorem 2.5.3 in [10]):

$$(2.10) \quad \begin{aligned} \mathbb{E}[\Delta_t(x, y)^2] &= \sigma_\xi^2 \sum_{k=0}^{\infty} \sum_{z \in \mathbb{Z}^d} (p^k(y, z) - p^k(x, z))^2 \\ &= 2\sigma_\xi^2 \sum_{k=0}^{\infty} (q^k(0, 0) - q^k(x - y, 0)) = 2\sigma_\xi^2 a(x - y) \end{aligned}$$

where we introduced the symmetric random walk kernel

$$(2.11) \quad q(x, y) = q(0, y - x) = \sum_{z \in \mathbb{Z}^d} p(0, z) p(x, y + z), \quad x, y \in \mathbb{Z}^d,$$

and its potential kernel

$$(2.12) \quad a(x) = \sum_{k=0}^{\infty} [q^k(0, 0) - q^k(x, 0)], \quad x \in \mathbb{Z}^d.$$

The strong aperiodicity assumption in (2.1) guarantees that  $\mathbb{Z}^d$  is the smallest subgroup that contains the support  $\{x : q(x) > 0\}$ . Convergence of (2.12) under this property is classical: see Section 28 in [34] for  $d = 1$  and Section 12 for  $d = 2$ . In  $d \geq 3$  convergence follows from the transience of the  $q$ -walk.

REMARK 2.2. The convergence issue here is real. Suppose  $p(-1) + p(1) = 1$  in  $d = 1$  so that  $q$  is supported on  $\{-2, 0, 2\}$  and assumption (2.1) is violated. Then from (2.9)

$$\Delta_t(0, 1) = \sum_{k=0}^{\infty} \sum_{z \in \mathbb{Z}} \xi_{t-k}(z) p^k(1, z) - \sum_{k=0}^{\infty} \sum_{z \in \mathbb{Z}} \xi_{t-k}(z) p^k(0, z)$$

splits into two independent sums because for a given  $(k, z)$  pair at most one of  $p^k(1, z)$  and  $p^k(0, z)$  is nonzero. That this series cannot converge even in distribution in  $d = 1$  can be shown with the characteristic function argument of the proof of Theorem 2.11(a) in Section 4.

One checks from the definition (2.9) that the variables  $\Delta_t(x, y)$  satisfy additivity (2.6) and the evolution equation

$$(2.13) \quad \Delta_{t+1}(x, y) = \sum_z p(0, z) \Delta_t(x + z, y + z) + \xi_{t+1}(y) - \xi_{t+1}(x), \quad x, y \in \mathbb{Z}^d, \quad t \in \mathbb{Z}.$$

Note that  $\Delta_t(x, y)$  is a function of  $\{\xi_s\}_{s:s \leq t}$  and thereby independent of  $\xi_{t+1}$ . Hence (2.13) describes a Markovian evolution.  $\{\Delta_t(x, y)\}$  is an increment process of a height process.

By additivity (2.6) it is enough to keep track of nearest-neighbor increments. Let us simplify the process to  $\eta_t = \{\eta_t(x)\}_{x \in \mathbb{Z}^d}$  with the vector-valued spin variable  $\eta_t(x) = (\eta_t(x - e_i, x))_{i \in [d]} \in \mathbb{R}^d$ . (The notation is  $[d] = \{1, 2, \dots, d\}$ .) The state space of the time- $t$  configuration  $\eta_t$  is  $(\mathbb{R}^d)^{\mathbb{Z}^d}$ . For  $y - x = e_i$  the evolution equation (2.7) specializes to

$$(2.14) \quad \eta_{t+1}(x - e_i, x) = \sum_z p(0, z) \eta_t(x + z - e_i, x + z) + \xi_{t+1}(x) - \xi_{t+1}(x - e_i),$$

for  $x \in \mathbb{Z}^d$  and  $i = 1, \dots, d$ . We now consider the  $(\mathbb{R}^d)^{\mathbb{Z}^d}$ -valued process  $\eta_t$  defined by (2.14), without imposing the additivity requirement (2.6).

Let us establish some standard notation and terminology. A generic element of the state space  $(\mathbb{R}^d)^{\mathbb{Z}^d}$  of process (2.14) is denoted by  $\eta = (\eta(x))_{x \in \mathbb{Z}^d}$  with  $\eta(x) = (\eta(x - e_i, x))_{i \in [d]} \in \mathbb{R}^d$ .  $\mathcal{M}_1 = \mathcal{M}_1((\mathbb{R}^d)^{\mathbb{Z}^d})$  is the space of probability measures on  $(\mathbb{R}^d)^{\mathbb{Z}^d}$ . A measure  $\mu \in \mathcal{M}_1$  is invariant for process (2.14) if equation (2.14) preserves this distribution: that is, if  $\eta_0 \sim \mu$  and noise  $\xi_1 = \{\xi_1(x)\}_{x \in \mathbb{Z}^d}$  is independent of  $\eta_0$ , then  $\eta_1$  defined by (2.14) satisfies again  $\eta_1 \sim \mu$ .  $\mathcal{I}$  denotes the convex set of probability measures in  $\mathcal{M}_1$  that are invariant for process (2.14).

Write  $\{\theta_{x,t}\}_{x \in \mathbb{Z}^d, t \in \mathbb{Z}}$  for shift mappings in space ( $x$ -index) and time ( $t$ -index). When only space or only time is shifted, the other index is set to zero. For example, for  $\eta \in (\mathbb{R}^d)^{\mathbb{Z}^d}$ ,  $(\theta_{a,0}\eta)(x - e_i, x) = \eta(x + a - e_i, x + a)$ . A probability measure  $\mu \in \mathcal{M}_1$  is (spatially) shift-invariant if  $\mu(\theta_{x,0}A) = \mu(A)$  for all Borel sets  $A \subseteq (\mathbb{R}^d)^{\mathbb{Z}^d}$  and  $x \in \mathbb{Z}^d$ . A Borel set  $B \subseteq (\mathbb{R}^d)^{\mathbb{Z}^d}$  is shift-invariant if  $\theta_{x,0}B = B \forall x \in \mathbb{Z}^d$ . A shift-invariant probability measure  $\mu \in \mathcal{M}_1$  is ergodic if  $\mu(B) \in \{0, 1\}$  for every invariant Borel set  $B$ . A shift-invariant probability measure  $\mu$  on  $(\mathbb{R}^d)^{\mathbb{Z}^d \times \mathbb{Z}}$  is ergodic under the individual space-time shift  $\theta_{z,s}$  if  $\mu(A) \in \{0, 1\}$  for every Borel set  $A$  that satisfies  $\theta_{z,s}A = A$ .

The Markov chain  $\eta_t$  defined by (2.14) without the additivity requirement (2.6) gives us a larger class of processes than the increment processes coming from height processes.

But we find that, under very lenient moment assumptions, all stationary  $\eta$ . processes are obtained by adding harmonic functions to  $\Delta$ . of (2.9).

The first theorem summarizes what was developed above. Set

$$(2.15) \quad \Delta_t(x) = (\Delta_t(x - e_i, x))_{i \in [d]} \in \mathbb{R}^d \quad \text{and} \quad \Delta_t = \{\Delta_t(x)\}_{x \in \mathbb{Z}^d} \in (\mathbb{R}^d)^{\mathbb{Z}^d}.$$

The invariance and ergodicity claim below follows from  $\Delta_t(x)(\xi) = \Delta_0(0)(\theta_{x,t}\xi)$ . Let  $\pi_0 \in \mathcal{I}$  denote the distribution of  $\Delta_t$ .

**THEOREM 2.3.** *Assume (2.1) and (2.4). Then the series in (2.9) converges for almost every  $\xi$  and in  $L^2(\mathbb{P})$ . For almost every  $\xi$  the variables  $\Delta_t(x, y)$  satisfy additivity (2.6) and the evolution equation (2.13). The process  $\Delta = \{\Delta_t(x)\}_{t \in \mathbb{Z}, x \in \mathbb{Z}^d}$  defined in (2.15) is a stationary version of the Markov chain (2.14) and is invariant and ergodic under each individual space-time shift mapping  $\theta_{x,t}$  for  $(x, t) \neq (0, 0)$ .*

Next we add harmonic functions to the variables  $\Delta_t(x - e_i, x)$  to produce invariant distributions with nonzero means and address uniqueness and convergence.

A function  $v : \mathbb{Z}^d \rightarrow \mathbb{R}$  is *harmonic* for transition probability kernel  $p$  if

$$(2.16) \quad \sum_{y \in \mathbb{Z}^d} p(x, y)v(y) = v(x) \quad \text{for all } x \in \mathbb{Z}^d.$$

Constants are the only bounded harmonic functions (Lemma A.1 in the appendix) but there can be many unbounded ones. Let  $\mathcal{H}_d$  denote the space of harmonic functions  $u : \mathbb{Z}^d \rightarrow \mathbb{R}^d$ , in other words, the space of vectors  $u = (u_1, \dots, u_d)$  of  $d$  real-valued harmonic functions  $u_i : \mathbb{Z}^d \rightarrow \mathbb{R}$ . Given  $u \in \mathcal{H}_d$ , define a process  $\Delta_t^u = u + \Delta_t$  by

$$(2.17) \quad \Delta_t^u(x - e_i, x) = u_i(x) + \Delta_t(x - e_i, x), \quad t \in \mathbb{Z}, x \in \mathbb{Z}^d, i \in [d],$$

where  $\Delta_t(x - e_i, x)$  is defined by (2.9).  $\Delta_t^u$  is another time-stationary and ergodic process whose evolution obeys (2.14). Let  $\pi_u \in \mathcal{I}$  denote the distribution of the configuration  $\Delta_t^u$ .

For nonconstant  $u$ ,  $\Delta_t^u$  may fail additivity (2.6). We are now considering the broader class of processes defined by (2.14), regardless of whether this process comes from a height process.

We record the invariant distributions  $\pi_u$  in the next theorem. That  $\pi_u$  is an *extreme point* of  $\mathcal{I}$  means that if  $\pi_u = b\mu + (1 - b)\nu$  for  $0 < b < 1$  and  $\mu, \nu \in \mathcal{I}$ , then  $\mu = \nu = \pi_u$ .

**THEOREM 2.4.** (Existence.) *Assume (2.1) and (2.4). Then the measures  $\{\pi_u : u \in \mathcal{H}_d\}$  are extreme points of  $\mathcal{I}$ . They are also the unique invariant distributions of minimal variance: for any  $\nu \in \mathcal{I}$  with finite variances,*

$$(2.18) \quad \text{Var}^\nu[\eta(x - e_i, x)] \geq \text{Var}^{\pi_0}[\eta(x - e_i, x)],$$

*and equality holds for all  $i \in [d]$  and  $x \in \mathbb{Z}^d$  iff  $\nu \in \{\pi_u : u \in \mathcal{H}_d\}$ .*

The next theorem shows that these are the unique extreme invariant distributions under a growth condition on the centered first moment of  $\eta(x)$ .

**THEOREM 2.5.** (Uniqueness.) *Assume (2.1) and (2.4). Let  $\nu \in \mathcal{I}$  satisfy these properties:  $E^\nu|\eta(x)| < \infty \forall x \in \mathbb{Z}^d$  and there exists  $u \in \mathcal{H}_d$  such that*

$$(2.19) \quad \lim_{r \rightarrow \infty} r^{-1/2} \cdot \max_{x \in \mathbb{Z}^d: |x| \leq r} E^\nu \{ |\eta(x) - u(x)| \} = 0.$$



Then (2.19) holds also if  $u(x)$  is replaced by  $\bar{u}(x) = E^\nu[\eta(x)]$ . There exists a probability measure  $\gamma$  on  $\mathbb{R}^d$  such that  $\nu = \int_{\mathbb{R}^d} \pi_{\bar{u}+\alpha} \gamma(d\alpha)$ .

Above  $\pi_{\bar{u}+\alpha}$  is the measure  $\pi_u$  for the harmonic function  $u(x) = \bar{u}(x) + \alpha$  in  $\mathcal{H}_d$ .

REMARK 2.6. The key to the proof of Theorem 2.5 is the local central limit theorem applied to the random walk  $p^t(0, x)$ . The factor  $r^{-1/2}$  in assumption (2.19) comes from a smoothness bound of Gamkrelidze [14, Theorem 4]:

$$(2.20) \quad \sum_{x \in \mathbb{Z}^d} \sum_{\ell=1}^d |p^t(0, x) - p^t(0, x - e_\ell)| \leq Ct^{-1/2} \quad \forall t \in \mathbb{Z}_+.$$

For an irreducible, aperiodic kernel  $p$  the same estimate is given in Prop. 2.4.2 of [22].

The third item is convergence to an invariant distribution. We currently have a result only when the centered initial distribution is of this type:

$$(2.21) \quad \begin{aligned} &\zeta = \{\zeta(x)\}_{x \in \mathbb{Z}^d} \text{ is an } (\mathbb{R}^d)^{\mathbb{Z}^d}\text{-valued random configuration whose distribution} \\ &\text{is invariant and ergodic under the spatial shift group } \{\theta_{x,0}\}_{x \in \mathbb{Z}^d}, \text{ and} \\ &\text{the } \mathbb{R}^d\text{-valued variable } \zeta(x) = (\zeta(x, i))_{i \in [d]} \text{ has mean } E\zeta(x) = 0. \end{aligned}$$

THEOREM 2.7. (Convergence.) Assume (2.1) and (2.4). Let  $\{\eta_t\}_{t \in \mathbb{Z}_+}$  be an  $(\mathbb{R}^d)^{\mathbb{Z}^d}$ -valued process whose evolution is defined by (2.14). Assume that the initial distribution of the process  $\eta_t$  is of the following form:  $\eta_0(x - e_i, x) = u_i(x) + \zeta(x, i)$  for  $x \in \mathbb{Z}^d$  and  $i = 1, \dots, d$  where  $u = (u_1, \dots, u_d) \in \mathcal{H}_d$  and  $\zeta$  is as in (2.21). Let  $\mu_t$  denote the distribution of  $\eta_t$ . Then, as  $t \rightarrow \infty$ ,  $\mu_t \rightarrow \pi_u$  weakly in  $\mathcal{M}_1$ .

REMARK 2.8 (Spatially invariant case). Let us spell out the most natural special case. Given a constant  $\alpha \in \mathbb{R}^d$ , there is a unique  $\pi_\alpha \in \mathcal{I}$  that is invariant and ergodic for the process  $\eta_t$  defined by (2.14), invariant and ergodic under the spatial shift group  $\{\theta_{x,0}\}_{x \in \mathbb{Z}^d}$ , and has mean  $E^{\pi_\alpha}[\eta(x)] = \alpha$ .  $\pi_\alpha$  is the distribution of the  $(\mathbb{R}^d)^{\mathbb{Z}^d}$ -valued random configuration  $\{\zeta(x)\}_{x \in \mathbb{Z}^d}$  defined by  $\zeta(x) = \alpha + (\Delta(x - e_i, x))_{i \in [d]}$ . Furthermore, the process (2.14) started with an arbitrary mean- $\alpha$  spatially invariant and ergodic initial distribution converges weakly to  $\pi_\alpha$ .

In the one-dimensional case that is the subject of the next section,  $\pi_0$  is the distribution of the sequence  $\{\Delta_0(x)\}_{x \in \mathbb{Z}}$  where  $\Delta_0(x) = \Delta_0(x - 1, x)$  is defined by the series (2.9). The next theorem collects properties of the shift-invariant covariance

$$V_0(x, y) = \mathbb{E}[\Delta_0(x)\Delta_0(y)] = E^{\pi_0}[\eta(x)\eta(y)] = E^{\pi_0}[\eta(0)\eta(y - x)] = V_0(0, y - x)$$

of  $\pi_0$  in  $d = 1$ . The span 1 assumption on  $p$  implies that the support of  $q$  cannot lie in a proper subgroup of  $\mathbb{Z}$ , and thereby the characteristic function  $\phi_q(\theta) = \sum_{x \in \mathbb{Z}} q(0, x)e^{i\theta x}$  equals 1 only at  $\theta \in 2\pi\mathbb{Z}$ .

THEOREM 2.9. Assume (2.1), (2.4), and  $d = 1$ . Then  $\exists$  constants  $0 < A, c < \infty$  such that  $|V_0(0, x)| \leq Ae^{-c|x|}$ . We have the identities

$$(2.22) \quad V_0(0, x) = \sigma_\xi^2[a(x - 1) + a(x + 1) - 2a(x)] = \frac{\sigma_\xi^2}{\pi} \int_{-\pi}^{\pi} \frac{1 - \cos \theta}{1 - \phi_q(\theta)} e^{i\theta x} d\theta, \quad x \in \mathbb{Z},$$



and

$$(2.23) \quad \sum_{x \in \mathbb{Z}} V_0(0, x) = \frac{\sigma_\xi^2}{\sigma_1^2}$$

where the series above is absolutely convergent.

The first equality in (2.22) comes from a series like (2.10) and the second equality is from p. 355 of Spitzer [34]. The rest is proved in Section 4 from properties of the kernel  $a(x)$ .

REMARK 2.10 ( $d = 1$  Gaussian case). Presently we cannot say more about the distribution  $\pi_0$ , except in the Gaussian case. If  $\{\xi_t(x)\}$  are centered Gaussian variables, then  $\pi_0$  is the mean zero Gaussian measure with covariance (2.22). Furthermore, if  $p(0, z) + p(0, z+1) = 1$  for some  $z \in \mathbb{Z}$ , then  $q$  is supported on  $\{-1, 0, 1\}$  and (2.22) shows that  $V_0(0, x) = 0$  for  $x \neq 0$ . In other words, the variables  $\{\Delta_0(x)\}$  are uncorrelated, and thereby in the Gaussian case they are i.i.d.

We turn briefly to the invariant distributions of the height process  $h_t$ . The key point is that in  $d \in \{1, 2\}$  there are none. By analogy with (2.9), define

$$(2.24) \quad \chi_t(x) = \sum_{k=0}^{\infty} \sum_{z \in \mathbb{Z}^d} \xi_{t-k}(z) p^k(x, z), \quad x \in \mathbb{Z}^d, t \in \mathbb{Z}.$$

Computation of the sum of variances on the right gives

$$\mathbb{E}[\chi_t(x)^2] = \sigma_\xi^2 \sum_{k=0}^{\infty} \sum_{z \in \mathbb{Z}^d} p^k(x, z)^2 = \sigma_\xi^2 \sum_{k=0}^{\infty} q^k(0, 0) = \sigma_\xi^2 G(0, 0)$$

where the Green function

$$(2.25) \quad G(x, y) = \sum_{k=0}^{\infty} q^k(x, y), \quad x, y \in \mathbb{Z}^d,$$

converges by transience in  $d \geq 3$ .

THEOREM 2.11. Assume (2.1) and (2.4). Consider the height process  $h_\bullet$  defined by (2.3).

- (a) In dimensions  $d = 1$  and  $d = 2$  this process has no invariant distributions.
- (b) In dimensions  $d \geq 3$  the series in (2.24) converges for almost every  $\xi$  and in  $L^2(\mathbb{P})$ . For almost every  $\xi$  the variables  $\chi_t(x)$  satisfy equation (2.3). The process  $\chi_\bullet = \{\chi_t(x)\}_{t \in \mathbb{Z}, x \in \mathbb{Z}^d}$  is a stationary version of the Markov chain (2.3) and is invariant and ergodic under each individual space-time shift mapping  $\theta_{x,t}$  for  $(x, t) \neq (0, 0)$ .

Part (b) of the theorem is clear. Part (a) is proved in Section 4.

The stationary height and increment processes (2.24) and (2.9) are obviously connected by  $\Delta_t(x, y) = \chi_t(y) - \chi_t(x)$ . In fact any bi-infinite process  $\{\eta_t(x, y)\}_{t \in \mathbb{Z}, x, y \in \mathbb{Z}^d}$  that satisfies additivity (2.6) and evolution (2.7)  $\forall t \in \mathbb{Z}$  comes from a bi-infinite height process  $\{h_t\}_{t \in \mathbb{Z}}$ , uniquely determined once a value  $h_0(0)$  is chosen: the forward process  $\{h_t\}_{t \in \mathbb{Z}_+}$  is determined by (2.5) and the evolution (2.3). We extend the height process backward in

time inductively. Assuming that  $\{h_s\}_{s \geq t}$  has been constructed, extend to time  $t-1$  by first defining the value

$$h_{t-1}(0) = h_t(z) - \sum_y p(z, y) \eta_{t-1}(0, y) - \xi_t(z)$$

where  $z \in \mathbb{Z}^d$  is any point, and then  $h_{t-1}(x) = h_{t-1}(0) + \eta_{t-1}(0, x)$  for all  $x \neq 0$ . That the definition of  $h_{t-1}(0)$  is independent of  $z$  is equivalent to (2.7). That  $h_t$  now comes from  $h_{t-1}$  by (2.3) is immediate from the definitions.

### 3. HEIGHT FLUCTUATIONS IN ONE DIMENSION

Restrict to dimension  $d = 1$ . Assume that the initial height function  $h_0 = \{h_0(x)\}_{x \in \mathbb{Z}}$  is normalized by  $h_0(0) = 0$  and that the distribution of the initial increment configuration  $\eta_0 = \{\eta_0(x) = h_0(x) - h_0(x-1)\}_{x \in \mathbb{Z}}$  is invariant under translations of the spatial index  $x$ . The mean, variance, and series of covariances of the initial increment variables are

$$(3.1) \quad \mu_0 = \mathbf{E}[\eta_0(x)], \quad \sigma_0^2 = \mathbf{Var}[\eta_0(x)], \quad \text{and} \quad \varrho_0 = \sum_{x \in \mathbb{Z}} \mathbf{Cov}[\eta_0(0), \eta_0(x)].$$

Our assumptions will guarantee absolute convergence of the series. Then  $\varrho_0 \geq 0$  because it is the limit of  $n^{-1} \mathbf{Var}[\eta_0(1) + \dots + \eta_0(n)]$ .

Let  $b = -v_1 = -\sum_x xp(0, x)$ . Scale space by  $\sqrt{n}$  and time by  $n$  and consider the scaled and centered space-time height process

$$(3.2) \quad \bar{h}_n(t, r) = n^{-1/4} \{h_{\lfloor nt \rfloor}(\lfloor r\sqrt{n} \rfloor + \lfloor ntb \rfloor) - \mu_0 r \sqrt{n}\}, \quad (t, r) \in \mathbb{R}_+ \times \mathbb{R}.$$

We prove that this process converges to a Gaussian process.

The limit process has three natural descriptions: as a Gaussian process, as a sum of two stochastic integrals, and as the solution of the stochastic heat equation with additive noise. The proof is based on the Gaussian process description. Denote the centered Gaussian p.d.f and c.d.f with variance  $\nu^2$  by

$$(3.3) \quad \varphi_{\nu^2}(x) = \frac{1}{\sqrt{2\pi\nu^2}} \exp\left(-\frac{x^2}{2\nu^2}\right) \quad \text{and} \quad \Phi_{\nu^2}(x) = \int_{-\infty}^x \varphi_{\nu^2}(y) dy,$$

and let

$$(3.4) \quad \Psi_{\nu^2}(x) = \nu^2 \varphi_{\nu^2}(x) - x(1 - \Phi_{\nu^2}(x)), \quad x \in \mathbb{R}.$$

Define two positive definite functions  $\Gamma_1$  and  $\Gamma_2$  for  $(s, q), (t, r) \in \mathbb{R}_+ \times \mathbb{R}$  by

$$(3.5) \quad \Gamma_1((s, q), (t, r)) = \Psi_{\sigma_1^2(t+s)}(r - q) - \Psi_{\sigma_1^2|t-s|}(r - q)$$

and

$$(3.6) \quad \Gamma_2((s, q), (t, r)) = \Psi_{\sigma_1^2 s}(-q) + \Psi_{\sigma_1^2 t}(r) - \Psi_{\sigma_1^2(t+s)}(r - q).$$

Let  $\{Z(t, r) : (t, r) \in \mathbb{R}_+ \times \mathbb{R}\}$  be the mean zero Gaussian process with covariance

$$(3.7) \quad E[Z(s, q)Z(t, r)] = \frac{\sigma_\xi^2}{\sigma_1^2} \Gamma_1((t, r), (s, q)) + \varrho_0 \Gamma_2((t, r), (s, q)).$$

Equivalently, this process is given by

$$(3.8) \quad Z(t, r) = \frac{\sigma_\xi}{\sigma_1} \iint_{[0, t] \times \mathbb{R}} \varphi_{\sigma_1^2(t-s)}(r-x) dW(s, x) + \sqrt{\varrho_0} \int_{\mathbb{R}} \varphi_{\sigma_1^2 t}(r-x) B(x) dx,$$

where  $\{W(t, r) : t \in \mathbb{R}_+, r \in \mathbb{R}\}$  is a two-parameter Brownian motion and  $\{B(r) : r \in \mathbb{R}\}$  a two-sided Brownian motion, and  $W$  and  $B$  are independent. In the fluctuation limit of  $\bar{h}_n$  described by (3.7) and (3.8), the first term comes from the fluctuations of  $\xi$  (the dynamics), and the second term comes from the fluctuations of  $\eta_0$  propagated by the dynamics.

$Z$  is also the unique mild solution [36] of the stochastic partial differential equation

$$(3.9) \quad \frac{\partial Z}{\partial t} = \frac{1}{2} \sigma_1^2 \frac{\partial^2 Z}{\partial r^2} + \frac{\sigma_\xi}{\sigma_1} \dot{W} \quad \text{on } \mathbb{R}_+ \times \mathbb{R}, \quad Z(0, r) = \sqrt{\varrho_0} B(r).$$

We consider three different hypotheses on the initial increments  $\{\eta_0(x)\}$ : (a) i.i.d., (b) strongly mixing, and (c) the invariant distribution  $\pi_0$  of Theorem 2.3, defined by

$$(3.10) \quad \eta_0(x) = \sum_{y \in \mathbb{Z}} \sum_{k=0}^{\infty} \xi_{-k}(y) [p^k(x, y) - p^k(x-1, y)], \quad x \in \mathbb{Z}.$$

Throughout the text these are referred to as cases (a), (b) and (c). In case (c) parameter  $\varrho_0$  disappears from the limit because  $\varrho_0 = \sigma_\xi^2 / \sigma_1^2$  (2.23).

Given two sub- $\sigma$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$  on a probability space  $(\Omega, \mathcal{F}, P)$ , let

$$(3.11) \quad \alpha(\mathcal{A}, \mathcal{B}) = \sup_{A \in \mathcal{A}, B \in \mathcal{B}} |P(A \cap B) - P(A)P(B)|.$$

For the initial increment sequence  $\{\eta(x)\}$  define  $\sigma$ -algebras  $\mathcal{F}_{m,n}^\eta = \sigma\{\eta(x) : m \leq x \leq n\}$ , and then the *strong mixing coefficients*

$$(3.12) \quad \alpha(n) = \sup_k \alpha(\mathcal{F}_{-\infty, k}^\eta, \mathcal{F}_{k+n, \infty}^\eta), \quad n \in \mathbb{Z}_+.$$

The sequence  $\{\eta(x)\}$  is *strongly mixing* if  $\alpha(n) \rightarrow 0$  as  $n \rightarrow \infty$ . See Bradley [5] for properties of these and other mixing coefficients.

Here is the convergence of finite-dimensional distributions.

**THEOREM 3.1.** *Assume  $d = 1$ , (2.1), (2.4), and  $\mathbb{E}[\xi_t(x)^4] < \infty$ . Assume that the initial increment sequence  $\{\eta_0(x)\}_{x \in \mathbb{Z}}$  satisfies one of the assumptions (a), (b), or (c):*

(a)  $\{\eta_0(x)\}_{x \in \mathbb{Z}}$  is an i.i.d. sequence and  $\mathbf{E}[\eta_0(x)^2] < \infty$ .

(b)  $\{\eta_0(x)\}_{x \in \mathbb{Z}}$  is a strongly mixing stationary sequence and  $\exists \delta > 0$  such that  $\mathbf{E}[|\eta_0(0)|^{2+\delta}] < \infty$  and the strong mixing coefficients in (3.12) satisfy

$$(3.13) \quad \sum_{j=0}^{\infty} (j+1)^{2/\delta} \alpha(j) < \infty.$$

(c)  $\{\eta_0(x)\}_{x \in \mathbb{Z}}$  has the distribution  $\pi_0$  of the sequence in (3.10).

Then,  $\varrho_0 = \sum_{x \in \mathbb{Z}} \mathbf{Cov}[\eta_0(0), \eta_0(x)] \geq 0$  is absolutely convergent. As  $n \rightarrow \infty$  the finite-dimensional distributions of the process  $\bar{h}_n$  of (3.2) converge weakly to those of the mean-zero Gaussian process  $Z$  with covariance (3.7).

The convergence in the theorem means that for any fixed  $N \in \mathbb{N}$  and  $(t_1, r_1), (t_2, r_2), \dots, (t_N, r_N) \in \mathbb{R}_+ \times \mathbb{R}$ , the weak convergence of  $\mathbb{R}^N$ -valued vectors holds: as  $n \rightarrow \infty$ ,

$$(3.14) \quad (\bar{h}_n(t_1, r_1), \bar{h}_n(t_2, r_2), \dots, \bar{h}_n(t_N, r_N)) \Rightarrow (Z(t_1, r_1), Z(t_2, r_2), \dots, Z(t_N, r_N)).$$

REMARK 3.2. Case (a) is stated only for completeness, we do not prove it. A natural question is whether case (c) is actually covered by case (b). The answer is affirmative in the Gaussian case where,  $\forall n \in \mathbb{N}$ ,  $\alpha(k) = O(|k|^{-n})$  as  $|k| \rightarrow \infty$ . See [38] for details.

REMARK 3.3 (Fractional Brownian motion). In case (c) of the theorem, or in general whenever  $\varrho_0 = \sigma_\xi^2 / \sigma_1^2$ , the limit of the time-indexed process  $\bar{h}_n(t, 0)$  is the Gaussian process with covariance

$$(3.15) \quad E[Z(s, 0)Z(t, 0)] = \frac{\sigma_\xi^2}{\sqrt{2\pi\sigma_1^2}}(s^{1/2} + t^{1/2} - |t - s|^{1/2})$$

which is a fractional Brownian motion with Hurst parameter  $1/4$ .

By strengthening the assumptions we upgrade the weak convergence  $\bar{h}_n \Rightarrow Z$  to process level on a compact time-space rectangle  $Q = [0, T] \times [-R, R]$ .  $Z$  is a continuous process on  $Q$ . The paths of  $\bar{h}_n$  lie in the space  $D_2$  of 2-parameter cadlag paths, defined precisely as follows. Given  $(t_0, r_0) \in Q$ , define four quadrants by

$$\begin{aligned} Q_{(t_0, r_0)}^1 &= \{(t, r) \in Q : t \geq t_0, r \geq r_0\}, & Q_{(t_0, r_0)}^2 &= \{(t, r) \in Q : t \geq t_0, r < r_0\}, \\ Q_{(t_0, r_0)}^3 &= \{(t, r) \in Q : t < t_0, r < r_0\}, & Q_{(t_0, r_0)}^4 &= \{(t, r) \in Q : t < t_0, r \geq r_0\}. \end{aligned}$$

Then the path space is defined by

$$\begin{aligned} D_2 &= \left\{ f : Q \rightarrow \mathbb{R} : \forall (t_0, r_0) \in Q, \lim_{\substack{(t, r) \in Q_{(t_0, r_0)}^i \\ (t, r) \rightarrow (t_0, r_0)}} f(t, r) \text{ exists for } i \in \{1, 2, 3, 4\} \right. \\ &\quad \left. \text{and } \lim_{\substack{(t, r) \in Q_{(t_0, r_0)}^1 \\ (t, r) \rightarrow (t_0, r_0)}} f(t, r) = f(t_0, r_0) \right\}. \end{aligned}$$

$D_2$  is separable and topologically complete under a Skorohod-type metric

$$d(f, g) = \inf_{\lambda} \max(\|f - g \circ \lambda\|_\infty, \|\lambda - \text{id}\|_\infty), \quad f, g \in D_2,$$

where  $\lambda(t, r) = (\lambda_1(t), \lambda_2(r))$  for strictly increasing, continuous bijections  $\lambda_1$  and  $\lambda_2$ . For details we refer to Bickel and Wichura [3].

THEOREM 3.4. Assume  $d = 1$ , (2.1), (2.4), and  $\mathbb{E}[\xi_t(x)^{12}] < \infty$ . Assume that the initial increment sequence  $\{\eta_0(x)\}_{x \in \mathbb{Z}}$  satisfies one of the assumptions (a), (b), or (c):

- (a)  $\{\eta_0(x)\}_{x \in \mathbb{Z}}$  is an i.i.d. sequence and  $\mathbf{E}[\eta_0(x)^{12}] < \infty$ .
- (b)  $\{\eta_0(x)\}_{x \in \mathbb{Z}}$  is a strongly mixing stationary sequence, and  $\exists \delta > 0$  such that  $\mathbb{E}[|\eta_0(0)|^{12+\delta}] < \infty$  and the strong mixing coefficients in (3.12) satisfy

$$(3.16) \quad \sum_{j=0}^{\infty} (j+1)^{10+132/\delta} \alpha(j) < \infty.$$

- (c)  $\{\eta_0(x)\}_{x \in \mathbb{Z}}$  has the distribution  $\pi_0$  of the sequence in (3.10).

Then on any rectangle  $Q = [0, T] \times [-R, R]$ ,  $\bar{h}_n \Rightarrow Z$  on path space  $D_2$ .

This completes the description of results and we turn to proofs.

#### 4. PROOFS FOR INVARIANT DISTRIBUTIONS

For  $x \in \mathbb{Z}^d$ ,  $i \in [d]$ , and  $s < t$  in  $\mathbb{Z}$  let

$$\Delta_{s,t}(x - e_i, x) = \sum_{k=s+1}^t \sum_y \xi_k(y) [p^{t-k}(x, y) - p^{t-k}(x - e_i, y)].$$

Then  $\Delta_{-\infty,t}$  is what we denoted by  $\Delta_t$  in (2.9). As  $t - s \rightarrow \infty$ , the random process above indexed by  $(x, i)$  converges weakly to the configuration  $\Delta_0 \sim \pi_0$ , by the convergence in (2.9). The basic evolution equation (2.14) can be written as

$$(4.1) \quad \eta_t(x - e_i, x) = \sum_y p^t(x, y) \eta_0(y - e_i, y) + \Delta_{0,t}(x - e_i, x), \quad t \geq 0.$$

The proof of Theorem 2.4 is contained in the next lemma.

LEMMA 4.1. *We have these properties of invariant distributions.*

(a) *Let  $\nu \in \mathcal{I}$  and assume that under  $\nu$  each variable  $\eta(x - e_i, x)$  has finite mean and variance. Then*

$$(4.2) \quad \text{Var}^\nu[\eta(x - e_i, x)] \geq \text{Var}^{\pi_0}[\eta(x - e_i, x)].$$

*Equality in (4.2) holds for all  $i \in [d]$  and  $x \in \mathbb{Z}^d$  iff  $\nu \in \{\pi_u : u \in \mathcal{H}_d\}$ .*

(b) *For each  $u \in \mathcal{H}_d$ ,  $\pi_u$  is an extreme point of  $\mathcal{I}$ .*

*Proof.* (a) When the process  $\eta_t \sim \nu$  in (4.1) is stationary, taking expectations on both sides shows that  $u_i(x) = E^\nu[\eta(x - e_i, x)]$  is harmonic. Then also the process  $\bar{\eta}_t(x - e_i, x) = \eta_t(x - e_i, x) - u_i(x)$  satisfies (4.1) and is stationary. By stationarity, by the independence of the terms on the right of (4.1), by time-shift-invariance of  $\xi$ , by the  $L^2(\mathbb{P})$ -convergence in (2.9), and finally by the definition of  $\pi_0$ :

$$(4.3) \quad \begin{aligned} \text{Var}^\nu[\eta(x - e_i, x)] &= \mathbf{E}^\nu[\bar{\eta}_t(x - e_i, x)^2] \\ &= \mathbf{E}^\nu \left[ \left( \sum_y p^t(x, y) \bar{\eta}_0(y - e_i, y) \right)^2 \right] + \mathbb{E}[\Delta_{0,t}(x - e_i, x)^2] \\ &\geq \mathbb{E}[\Delta_{-t,0}(x - e_i, x)^2] \xrightarrow{t \rightarrow \infty} \mathbb{E}[\Delta_0(x - e_i, x)^2] = \text{Var}^{\pi_0}[\eta(x - e_i, x)]. \end{aligned}$$

This gives the inequality claimed in part (a).

Pick  $x_1, \dots, x_N \in \mathbb{Z}^d$ ,  $i_1, \dots, i_N \in [d]$ , and  $\alpha_1, \dots, \alpha_N \in \mathbb{R}$ , and repeat (4.1) for linear combinations:

$$(4.4) \quad \begin{aligned} \sum_{\ell=1}^N \alpha_\ell \bar{\eta}_t(x_\ell - e_{i_\ell}, x_\ell) &= \sum_{\ell=1}^N \alpha_\ell \sum_y p^t(x_\ell, y) \bar{\eta}_0(y - e_{i_\ell}, y) \\ &\quad + \sum_{\ell=1}^N \alpha_\ell \Delta_{0,t}(x_\ell - e_{i_\ell}, x_\ell) = S_t + D_t. \end{aligned}$$

Equality in (4.2)  $\forall i \in [d], x \in \mathbb{Z}^d$  and (4.3) imply that  $E^\nu[S_t^2] \rightarrow 0$ . Consequently, on the right-hand side of (4.4), the first sum vanishes and the second sum converges in distribution to  $\sum_\ell \alpha_\ell \Delta_0(x_\ell - e_{i_\ell}, x_\ell)$ . This implies that under  $\nu$ , the configuration  $\bar{\eta}$  has the distribution of the configuration  $\Delta_0$ , which says that  $\nu$  is among the distributions  $\{\pi_u : u \in \mathcal{H}_d\}$ .

(b) To get a contradiction, suppose  $\pi_u = \beta\nu^1 + (1 - \beta)\nu^0$  for  $\beta \in (0, 1)$  and  $\nu^1, \nu^0 \in \mathcal{I}$ . Let  $\eta^\ell$  denote the configuration under  $\nu^\ell$ . Let  $J$  be a Bernoulli( $\beta$ ) variable:  $P(J = 1) = \beta = 1 - P(J = 0)$ . Then  $\nu = \beta\nu^1 + (1 - \beta)\nu^0$  is the distribution of the configuration  $\eta^J$ , and by the definition of  $\pi_u$ , the assumption  $\pi_u = \nu$  implies the distributional equality

$$(4.5) \quad u_i(x) + \Delta_0(x - e_i, x) \stackrel{d}{=} \eta^J(x - e_i, x), \quad x \in \mathbb{Z}^d, i \in [d].$$

It follows that  $\eta^1$  and  $\eta^0$  have finite means  $u_i^\ell(x) = E^{\nu^\ell}[\eta^\ell(x - e_i, x)]$  and variances, and

$$(4.6) \quad u_i(x) = E^\nu[\eta^J(x - e_i, x)] = \beta u_i^1(x) + (1 - \beta)u_i^0(x).$$

From (4.5), by expanding the variance, and then by (4.2),

$$\begin{aligned} \text{Var}[\Delta_0(x - e_i, x)] &= E^\nu[(\eta^J(x - e_i, x) - u_i(x))^2] \\ &= \beta E^{\nu^1}[(\eta^1(x - e_i, x) - u_i^1(x) + (1 - \beta)\{u_i^1(x) - u_i^0(x)\})^2] \\ &\quad + (1 - \beta) E^{\nu^0}[(\eta^0(x - e_i, x) - u_i^0(x) - \beta\{u_i^1(x) - u_i^0(x)\})^2] \\ &= \beta \text{Var}^{\nu^1}[\eta^1(x - e_i, x)] + (1 - \beta) \text{Var}^{\nu^0}[\eta^0(x - e_i, x)] + \beta(1 - \beta)\{u_i^1(x) - u_i^0(x)\}^2 \\ &\geq \text{Var}[\Delta_0(x - e_i, x)] + \beta(1 - \beta)\{u_i^1(x) - u_i^0(x)\}^2. \end{aligned}$$

This forces first  $u^1 = u^2$ , so by (4.6)  $u = u^1 = u^0$ . Second, we get equality of the variances

$$\text{Var}^{\nu^1}[\eta^1(x - e_i, x)] = \text{Var}^{\nu^0}[\eta^0(x - e_i, x)] = \text{Var}[\Delta_0(x - e_i, x)]$$

which by part (a) forces  $\nu^\ell = \pi_u$  which equals  $\pi_u$ . □

We prove Theorem 2.7 next because it will be helpful for a later proof.

*Proof of Theorem 2.7.* By (4.1), the assumption on  $\eta_0$ , and harmonicity,

$$(4.7) \quad \begin{aligned} \eta_t(x - e_i, x) &= \sum_y p^t(x, y) u_i(y) + \sum_y p^t(x, y) \zeta(y, i) + \Delta_{0,t}(x - e_i, x) \\ &= u_i(x) + \sum_y p^t(x, y) \zeta(y, i) + \Delta_{0,t}(x - e_i, x). \end{aligned}$$

The terms on the right are independent. As  $t \rightarrow \infty$ , the second term on the right converges in  $L^1$  to the constant  $E\zeta(0, i) = 0$  by Lemma A.2 from the appendix. As a process indexed by  $(x, i)$ , the last term converges in distribution to  $\Delta_0$ . Since the configuration  $\{u_i(x) + \Delta_0(x - e_i, x)\}_{i \in [d], x \in \mathbb{Z}^d}$  has distribution  $\pi_u$ , we have shown that  $\eta_t$  converges weakly to  $\pi_u$ . □

For a probability measure  $\nu \in \mathcal{M}_1$ , let

$$A(\nu, r) = \max_{x \in \mathbb{Z}^d: |x| \leq r} E^\nu |\eta(x)|.$$

For configurations  $\eta \in (\mathbb{R}^d)^{\mathbb{Z}^d}$ , let

$$(4.8) \quad g_t(\eta, x, i) = \sum_y p^t(x, y) \eta(y - e_i, y).$$

LEMMA 4.2. *Let  $\nu \in \mathcal{M}_1$  satisfy  $r^{-1/2}A(\nu, r) \rightarrow 0$  as  $r \rightarrow \infty$ . Then for  $x \in \mathbb{Z}^d$  and  $i, j \in [d]$ ,*

$$\lim_{t \rightarrow \infty} E^\nu |g_t(\eta, x, i) - g_t(\eta, x - e_j, i)| = 0.$$

*Proof.* By Theorem 4 of Gamkrelidze [14] (see also the next to last paragraph of [14]), under assumptions (2.1) on the kernel, there is a constant  $C < \infty$  such that

$$(4.9) \quad \sum_{x \in \mathbb{Z}^d} \sum_{\ell=1}^d |p^t(0, x) - p^t(0, x - e_\ell)| \leq Ct^{-1/2} \quad \forall t \in \mathbb{Z}_+.$$

By (2.1) kernel  $p$  has finite range  $M$ . So by (4.9),

$$\begin{aligned} E^\nu |g_t(\eta, x, i) - g_t(\eta, x - e_j, i)| &\leq \sum_{y: |y| \leq Mt+1} |p^t(0, y) - p^t(0, y + e_j)| \cdot E^\nu |\eta(x + y)| \\ &\leq Ct^{-1/2} A(\nu, Mt + |x| + 1) \end{aligned}$$

and by the assumption the last quantity vanishes as  $t \rightarrow \infty$ .  $\square$

For  $\alpha \in \mathbb{R}^d$  let  $\pi_\alpha = \pi_u$  for the constant function  $u(x) = \alpha$ .

LEMMA 4.3. *Let  $\nu \in \mathcal{I}$  satisfy  $r^{-1/2}A(\nu, r) \rightarrow 0$  as  $r \rightarrow \infty$ . Then there exists a probability measure  $\gamma$  on  $\mathbb{R}^d$  such that  $\nu = \int \pi_\alpha \gamma(d\alpha)$ .*

*Proof.* Using (4.8) write the stationary evolution (4.1) for  $\eta_t \sim \nu$  as

$$(4.10) \quad \eta_t(x - e_i, x) = g_t(\eta_0, x, i) + \Delta_{0,t}(x - e_i, x).$$

Pick lattice points  $x_k \in \mathbb{Z}^d$ , indices  $i_k \in [d]$ , and reals  $\alpha_k$ , and consider the distributions of the linear combinations

$$\sum_{k=1}^N \alpha_k \eta_t(x_k - e_{i_k}, x_k) = \sum_{k=1}^N \alpha_k g_t(\eta_0, x_k, i_k) + \sum_{k=1}^N \alpha_k \Delta_{0,t}(x_k - e_{i_k}, x_k) = S_t + D_t$$

and the same thing shifted by  $-e_j$ :

$$\begin{aligned} \sum_{k=1}^N \alpha_k \eta_t(x_k - e_j - e_{i_k}, x_k - e_j) &= \sum_{k=1}^N \alpha_k g_t(\eta_0, x_k - e_j, i_k) \\ &\quad + \sum_{k=1}^N \alpha_k \Delta_{0,t}(x_k - e_j - e_{i_k}, x_k - e_j) = \tilde{S}_t + \tilde{D}_t. \end{aligned}$$



Compare  $\nu$  and its shift by  $-e_j$  through characteristic functions. Use time invariance, the triangle inequality, and  $|e^{ia}| \leq 1$  and  $|e^{ia} - e^{ib}| \leq |a - b|$  for real  $a, b$ .

$$\begin{aligned}
& \left| E^\nu[e^{\iota \sum_{k=1}^N \alpha_k \eta(x_k - e_{i_k}, x_k)}] - E^\nu[e^{\iota \sum_{k=1}^N \alpha_k \eta(x_k - e_j - e_{i_k}, x_k - e_j)}] \right| \\
&= \left| \mathbf{E}^\nu[e^{\iota \sum_{k=1}^N \alpha_k \eta_t(x_k - e_{i_k}, x_k)}] - \mathbf{E}^\nu[e^{\iota \sum_{k=1}^N \alpha_k \eta_t(x_k - e_j - e_{i_k}, x_k - e_j)}] \right| \\
&= \left| \mathbf{E}^\nu[e^{\iota S_t}] \mathbb{E}[e^{\iota D_t}] - \mathbf{E}^\nu[e^{\iota \tilde{S}_t}] \mathbb{E}[e^{\iota \tilde{D}_t}] \right| \\
&\leq \left| \mathbf{E}^\nu[e^{\iota S_t}] - \mathbf{E}^\nu[e^{\iota \tilde{S}_t}] \right| + \left| \mathbb{E}[e^{\iota D_t}] - \mathbb{E}[e^{\iota \tilde{D}_t}] \right| \\
&\leq \mathbf{E}^\nu|S_t - \tilde{S}_t| + \left| \mathbb{E}[e^{\iota D_t}] - \mathbb{E}[e^{\iota \tilde{D}_t}] \right|.
\end{aligned}$$

The last line above vanishes as  $t \rightarrow \infty$  by Lemma 4.2 and because process  $\Delta_{0,t}$  converges weakly to process  $\Delta_0$  which is invariant under spatial shifts. We have now shown  $\nu$  invariant under spatial shifts.

Let  $\nu = \int \mu \Gamma(d\mu)$  be the ergodic decomposition of  $\nu$ .  $\Gamma$  is a probability measure supported on probability measures  $\mu \in \mathcal{M}_1$  that are invariant and ergodic under the spatial shift group and that have a finite mean  $\alpha(\mu) = E^\mu[\eta(x)]$ .

Let  $f$  be a bounded continuous function on  $(\mathbb{R}^d)^{\mathbb{Z}^d}$ . By invariance of  $\nu$  and Theorem 2.7,

$$\int f d\nu = \mathbf{E}^\nu[f(\eta_t)] = \int \mathbf{E}^\mu[f(\eta_t)] \Gamma(d\mu) \xrightarrow{t \rightarrow \infty} \int E^{\pi_{\alpha(\mu)}}(f) \Gamma(d\mu) = \int E^{\pi_\alpha}(f) \gamma(d\alpha)$$

where  $\gamma$  is the distribution of the mean of  $\mu$  under  $\Gamma(d\mu)$ .  $\square$

*Proof of Theorem 2.5.* By harmonicity of  $u$ , the process  $\tilde{\eta}_t(x - e_i, x) = \eta_t(x - e_i, x) - u_i(x)$  is also time-stationary. By assumption (2.19) its marginal distribution  $\tilde{\nu} \in \mathcal{I}$  satisfies the growth bound  $r^{-1/2}A(\tilde{\nu}, r) \rightarrow 0$ . Lemma 4.3 applied to  $\tilde{\nu}$  gives the distributional identity  $\tilde{\eta}(x) \stackrel{d}{=} X + \Delta_0(x)$  where  $\tilde{\eta} \sim \tilde{\nu}$  and  $X \in \mathbb{R}^d$  is a  $\tilde{\gamma}$ -distributed random vector independent of  $\Delta_0$ . Consequently  $\bar{u}(x) = E^\nu[\eta(x)] = u(x) + E^{\tilde{\gamma}}[X]$ , and the claim about substituting  $\bar{u}(x)$  into (2.19) follows.

An application of Lemma 4.3 to the distribution  $\bar{\nu}$  of the configuration  $\bar{\eta}_t(x) = \eta_t(x) - \bar{u}(x)$  gives a random  $Y \in \mathbb{R}^d$  such that  $\eta \sim \nu$  satisfies  $\{\eta(x)\}_{x \in \mathbb{Z}^d} \stackrel{d}{=} \{\bar{u}(x) + Y + \Delta_0(x)\}_{x \in \mathbb{Z}^d}$ . With  $Y \sim \gamma$ , this is the same as  $\nu = \int \pi_{\bar{u} + \alpha} \gamma(d\alpha)$ .  $\square$

*Proof of Theorem 2.9.* Identity (2.22) says that  $V_0(0, x)$  is a Fourier coefficient of (a constant multiple of) the function  $f(\theta) = \frac{1 - \cos \theta}{1 - \phi_q(\theta)}$  on  $[-\pi, \pi]$ . Extend  $f$  from  $[-\pi, \pi]$  to a meromorphic function  $f(z) = \frac{1 - \cos z}{1 - \phi_q(z)}$  on the complex plane. The only pole in some neighborhood of  $[-\pi, \pi]$  is  $z = 0$ . Expansions give  $f(z) = \frac{z^2/2 + O(z^4)}{\sigma_1^2 z^2 + O(z^4)}$  and we see that  $z = 0$  is a removable singularity. Consequently  $f$  is analytic on  $[-\pi, \pi]$  and its Fourier coefficients decay exponentially (Prop. 1.2.20 on p. 20 of [28]).

Identity (2.23) now follows from

$$(4.11) \quad \lim_{x \rightarrow \pm \infty} [a(x + k) - a(x)] = \pm \frac{k}{2\sigma_1^2}$$

(P29.2 on p. 354 of [34]). Note that the variance of the  $q$ -kernel is  $2\sigma_1^2$  (2.2).  $\square$

*Proof of part (a) of Theorem 2.11.* By square-integrability (2.4) and Theorem 3.3.8 in [10], the characteristic function  $\varphi(\alpha) = \mathbb{E}(e^{\iota\alpha\xi_0(0)})$  satisfies

$$\varphi(\alpha) = 1 - \frac{1}{2}\alpha^2\sigma_\xi^2 + o(\alpha^2) \quad \text{as } \alpha \rightarrow 0.$$

Suppose  $\{h_t\}_{t \in \mathbb{Z}_+}$  is a time-stationary  $\mathbb{R}^{\mathbb{Z}^d}$ -valued Markov chain that satisfies (2.3). Iterating (2.3) gives

$$(4.12) \quad h_t(x) = \sum_{y \in \mathbb{Z}^d} p^t(x, y) h_0(y) + \sum_{k=0}^{t-1} \sum_{y \in \mathbb{Z}^d} p^k(x, y) \xi_{t-k}(y).$$

On the right-hand side the initial profile  $h_0$  is independent of the i.i.d. variables  $\{\xi_j(y)\}$ . By this independence, by the fact that characteristic functions are bounded in absolute value by 1, and by the invariance  $h_0 \stackrel{d}{=} h_t$ , we have, for any  $0 < s < t$ ,

$$\begin{aligned} |\mathbb{E}[e^{\iota\alpha h_0(x)}]| &\leq |\mathbb{E}[e^{\iota\alpha \sum_{k=s}^{t-1} \sum_{y \in \mathbb{Z}^d} p^k(x, y) \xi_{t-k}(y)}]| = \prod_{k=s}^{t-1} \prod_{y \in \mathbb{Z}^d} |\varphi(\alpha p^k(x, y))| \\ &= \prod_{k=s}^{t-1} \prod_{y \in \mathbb{Z}^d} \left(1 - \frac{1}{2}\alpha^2\sigma_\xi^2 (p^k(x, y))^2 (1 + o(1))\right) \\ &\leq \exp\left\{-\frac{1}{2}\alpha^2\sigma_\xi^2 (1 + o(1)) \sum_{k=s}^{t-1} \sum_{y \in \mathbb{Z}^d} (p^k(x, y))^2\right\} \leq \exp\left\{-c\alpha^2 \sum_{k=s}^{t-1} q^k(0, 0)\right\} \end{aligned}$$

with a constant  $c > 0$ . The second equality above is justified by fixing  $s$  large enough and by the dimension-independent uniform bound  $p^k(x, y) \leq Cs^{-1/2}$  for  $k \geq s$  (P7.6 on p. 72 in [34]). In dimensions  $d \in \{1, 2\}$  the  $q$ -walk is recurrent (T8.1 on p. 83 in [34]), and consequently taking  $t \rightarrow \infty$  above gives  $\mathbb{E}[e^{\iota\alpha h_0(x)}] = 0$  for  $\alpha \neq 0$ . This contradiction with the continuity of a characteristic function at  $\alpha = 0$  shows that in  $d \in \{1, 2\}$  there can exist no time-stationary height process.  $\square$

## 5. HEIGHT FLUCTUATIONS: LIMITS OF FINITE-DIMENSIONAL DISTRIBUTIONS

Let  $\{X_t^i\}_{t \in \mathbb{Z}_+}$  denote a random walk on  $\mathbb{Z}$  with initial point  $X_0^i = i$  and transition probability  $p(x, y)$  of (2.1), and let  $\{Y_t^i\}_{t \in \mathbb{Z}_+}$  similarly denote a random walk on  $\mathbb{Z}$  that uses transition  $q(x, y)$  of (2.11). Equivalently,  $Y_t^i = \tilde{X}_t^i - X_t^0$  for two independent  $p$ -walks  $\tilde{X}_t^i$  and  $X_t^0$ . Under assumption (2.1), the  $q$ -walk also has span 1.

Probabilities and expectations of these walks are denoted by  $P$  and  $E$ , always taken under fixed  $\eta_0$  and  $\xi$ . In particular, we have the following “dual” representation of the harness process: for  $t \in \mathbb{Z}_+$  and  $i \in \mathbb{Z}$ ,

$$(5.1) \quad h_t(i) = E[h_0(X_t^i)] + \sum_{k=1}^t E[\xi_k(X_{t-k}^i)],$$

where we emphasize that  $h_0$  and  $\xi$  have *not* been averaged over on the right because the  $E$  acts only on the walk  $X_t^i$ .

This lemma is a consequence of the local CLT (Theorem 3.5.2 in [10]) and will be used several times in the sequel.

LEMMA 5.1. *For a mean 0, span 1 random walk  $S_n$  on  $\mathbb{Z}$  with finite variance  $\sigma^2$ ,  $a \in \mathbb{R}$ , and points  $a_n \in \mathbb{Z}$  such that  $\lim_{n \rightarrow \infty} a_n/\sqrt{n} = a$ , we have*

$$(5.2) \quad \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \sum_{k=0}^{\lfloor nt \rfloor - 1} P(S_k = a_n) = \frac{1}{\sigma^2} \int_0^{\sigma^2 t} \frac{1}{\sqrt{2\pi v}} \exp\left(-\frac{a^2}{2v}\right) dv.$$

The analysis of height fluctuations begins with a decomposition of the scaled height function as

$$(5.3) \quad \bar{h}_n(t, r) = \mu_0 \bar{H}_n(t, r) + \bar{F}_n(t, r) + \bar{S}_n(t, r)$$

where, with  $y(n) = \lfloor nt \rfloor + \lfloor r\sqrt{n} \rfloor$ ,

$$(5.4) \quad \bar{H}_n(t, r) = n^{-1/4} (E[X_{\lfloor nt \rfloor}^{y(n)}] - r\sqrt{n}),$$

$$(5.5) \quad \bar{F}_n(t, r) = n^{-1/4} \sum_{k=1}^{\lfloor nt \rfloor} \sum_{x \in \mathbb{Z}} \xi_k(x) P(X_{\lfloor nt \rfloor - k}^{y(n)} = x) = n^{-1/4} \sum_{k=1}^{\lfloor nt \rfloor} E[\xi_k(X_{\lfloor nt \rfloor - k}^{y(n)})],$$

$$(5.6) \quad \bar{S}_n(t, r) = n^{-1/4} \sum_{i \in \mathbb{Z}} (\eta_0(i) - \mu_0) \left\{ \mathbf{1}_{\{i > 0\}} P(X_{\lfloor nt \rfloor}^{y(n)} \geq i) - \mathbf{1}_{\{i \leq 0\}} P(X_{\lfloor nt \rfloor}^{y(n)} < i) \right\}.$$

Recall that  $b = -v_1 = -E[X_1^0]$ . (5.3) follows from the random walk representation

$$(5.7) \quad \bar{h}_n(t, r) = n^{-1/4} \left\{ E[h_0(X_{\lfloor nt \rfloor}^{y(n)})] + \sum_{k=1}^{\lfloor nt \rfloor} E[\xi_k(X_{\lfloor nt \rfloor - k}^{y(n)})] - \mu_0 r \sqrt{n} \right\}$$

from (5.1),  $h_0(0) = 0$ ,  $\eta_0(k) = h_0(k) - h_0(k-1)$ , and from

$$\begin{aligned} & E[h_0(X_{\lfloor nt \rfloor}^{y(n)})] - \mu_0 r \sqrt{n} \\ &= E \left[ \mathbf{1}_{\{X_{\lfloor nt \rfloor}^{y(n)} > 0\}} \sum_{i=1}^{X_{\lfloor nt \rfloor}^{y(n)}} \eta_0(i) - \mathbf{1}_{\{X_{\lfloor nt \rfloor}^{y(n)} \leq 0\}} \sum_{i=X_{\lfloor nt \rfloor}^{y(n)}+1}^0 \eta_0(i) \right] - \mu_0 r \sqrt{n} \\ &= \sum_{i > 0} \eta_0(i) P(X_{\lfloor nt \rfloor}^{y(n)} \geq i) - \sum_{i \leq 0} \eta_0(i) P(X_{\lfloor nt \rfloor}^{y(n)} < i) - \mu_0 r \sqrt{n} \\ &= n^{1/4} \mu_0 \bar{H}_n(t, r) + n^{1/4} \bar{S}_n(t, r). \end{aligned}$$

The three terms in (5.3) will be treated separately both for the convergence of finite-dimensional distributions (this section) and process-level tightness (Section 6).

Begin by observing that the first term  $\mu_0 \bar{H}_n(t, r)$  on the right of (5.3) is irrelevant: since  $E(X_{\lfloor nt \rfloor}^{y(n)}) = v_1 \lfloor nt \rfloor + y(n) = -b \lfloor nt \rfloor + \lfloor nt \rfloor + \lfloor r\sqrt{n} \rfloor$ , we have  $\bar{H}_n(t, r) = O(n^{-1/4})$  and  $\lim_{n \rightarrow \infty} \mu_0 \bar{H}_n(t, r) = 0$ , uniformly over  $(t, r)$ .

Next note that  $\bar{F}_n$  and  $\bar{S}_n$  are independent.  $\bar{F}_n$  depends only on  $\xi$  and, in the limit, furnishes the first term in (3.7) and (3.8).  $\bar{S}_n$  depends only on  $\eta_0$  and gives in the limit

the second term in (3.7) and (3.8). The weak limits of  $\overline{F}_n$  and  $\overline{S}_n$  are treated separately in Propositions 5.2 and 5.3.

We begin with  $\overline{F}_n$ . Let  $\{F(t, r) : t \in \mathbb{R}_+, r \in \mathbb{R}\}$  be a mean-zero Gaussian process with covariance

$$(5.8) \quad E[F(t, r)F(s, q)] = \frac{\sigma_\xi^2}{\sigma_1^2} \Gamma_1((t, r), (s, q)).$$

Here are alternative expressions for  $\Gamma_1$  [33, Chapter 2]:

$$(5.9) \quad \Gamma_1((s, q), (t, r)) = \int_{-\infty}^{\infty} [P(B_{\sigma_1^2 s} \leq q - x)P(B_{\sigma_1^2 t} > r - x) - P(B_{\sigma_1^2 s} \leq q - x, B_{\sigma_1^2 t} > r - x)] dx,$$

where  $B_t$  is a standard 1-dimensional Brownian motion, and

$$(5.10) \quad \Gamma_1((s, q), (t, r)) = \frac{1}{2} \int_{\sigma_1^2 |t-s|}^{\sigma_1^2(t+s)} \frac{1}{\sqrt{2\pi v}} \exp\left\{-\frac{1}{2v}(r-q)^2\right\} dv.$$

PROPOSITION 5.2. Assume (2.4) and  $\mathbb{E}[\xi_t(x)^4] < \infty$ . Then, as  $n \rightarrow \infty$ , the finite-dimensional distributions of the process  $\overline{F}_n$  converge weakly to those of  $F$ .

*Proof.* First we argue the convergence of the covariance of  $\overline{F}_n$ . Let  $X_{\bullet}^{\lfloor nt \rfloor + \lfloor r\sqrt{n} \rfloor}$  and  $X_{\bullet}^{\lfloor ns \rfloor + \lfloor q\sqrt{n} \rfloor}$  denote two independent random walks with transition probability  $p$  (even if  $(t, r)$  and  $(s, q)$  should happen to coincide).

For  $s = t$  and  $r, q \in \mathbb{R}$ , set  $x_n = \lfloor r\sqrt{n} \rfloor - \lfloor q\sqrt{n} \rfloor$ . Then,

$$\begin{aligned} \mathbb{E}[\overline{F}_n(t, r)\overline{F}_n(t, q)] &= n^{-1/2} \sum_{k=1}^{\lfloor nt \rfloor} \sum_{x \in \mathbb{Z}} \sigma_\xi^2 P(X_{\lfloor nt \rfloor - k}^{\lfloor nt \rfloor + \lfloor r\sqrt{n} \rfloor} = x) P(X_{\lfloor nt \rfloor - k}^{\lfloor nt \rfloor + \lfloor q\sqrt{n} \rfloor} = x) \\ &= n^{-1/2} \sigma_\xi^2 \sum_{k=1}^{\lfloor nt \rfloor} q^{\lfloor nt \rfloor - k}(x_n, 0) = n^{-1/2} \sigma_\xi^2 \sum_{k=0}^{\lfloor nt \rfloor - 1} q^k(0, x_n) \\ &\xrightarrow{n \rightarrow \infty} \frac{\sigma_\xi^2}{2\sigma_1^2} \int_0^{2\sigma_1^2 t} \frac{1}{\sqrt{2\pi v}} \exp\left\{-\frac{(r-q)^2}{2v}\right\} dv = \frac{\sigma_\xi^2}{\sigma_1^2} \Gamma_1((t, q), (t, r)). \end{aligned}$$

The limit came from (5.2).

Next take  $s < t$  and set  $X'_n = X_{[nt]-[ns]}^{[ntb]+[r\sqrt{n}]} - [nsb] - [q\sqrt{n}]$ . By the Markov property,

$$\begin{aligned}
\mathbb{E}[\bar{F}_n(t, r)\bar{F}_n(s, q)] &= n^{-1/2}\sigma_\xi^2 \sum_{k=1}^{[ns]} P(X_{[nt]-k}^{[ntb]+[r\sqrt{n}]} - X_{[ns]-k}^{[nsb]+[q\sqrt{n}]} = 0) \\
&= n^{-1/2}\sigma_\xi^2 E\left[\sum_{k=1}^{[ns]} P(X_{[nt]-k}^{[ntb]+[r\sqrt{n}]} - X_{[ns]-k}^{[nsb]+[q\sqrt{n}]} = 0 \mid X_{[nt]-[ns]}^{[ntb]+[r\sqrt{n}]})\right] \\
&= n^{-1/2}\sigma_\xi^2 E\left[\sum_{k=1}^{[ns]} q^{[ns]-k} \left(X_{[nt]-[ns]}^{[ntb]+[r\sqrt{n}]} - [nsb] - [q\sqrt{n}], 0\right)\right] \\
&= n^{-1/2}\sigma_\xi^2 \sum_{k=0}^{[ns]-1} E[q^k(0, X'_n)].
\end{aligned}$$

We use again Lemma 5.1 to derive the limit. By the CLT,  $n^{-1/2}X'_n \Rightarrow B_{\sigma_1^2|t-s|} + (r - q)$ . By a standard construction (Theorem 3.2.2 in [10]), we can find random variables  $\hat{X}_n \stackrel{d}{=} X'_n$  such that  $n^{-1/2}\hat{X}_n \rightarrow B_{\sigma_1^2|t-s|} + (r - q)$  almost surely. Then by (5.2),

$$\lim_{n \rightarrow \infty} n^{-1/2} \sum_{k=0}^{[ns]-1} q^k(0, \hat{X}_n) = \frac{1}{2\sigma_1^2} \int_0^{2\sigma_1^2 s} \frac{1}{\sqrt{2\pi v}} \exp\left\{-\frac{(B_{\sigma_1^2|t-s|} + r - q)^2}{2v}\right\} dv \quad \text{a.s.}$$

By the uniform bound  $q^k(0, y) \leq Ck^{-1/2}$  we can apply dominated convergence.

$$\begin{aligned}
(5.11) \quad \lim_{n \rightarrow \infty} \mathbb{E}[\bar{F}_n(t, r)\bar{F}_n(s, q)] &= \lim_{n \rightarrow \infty} n^{-1/2}\sigma_\xi^2 E\left[\sum_{k=0}^{[ns]-1} q^k(0, \hat{X}_n)\right] \\
&= \sigma_\xi^2 E\left[\frac{1}{2\sigma_1^2} \int_0^{2\sigma_1^2 s} \frac{1}{\sqrt{2\pi v}} \exp\left\{-\frac{(B_{\sigma_1^2|t-s|} + r - q)^2}{2v}\right\} dv\right] \\
&= \frac{\sigma_\xi^2}{2\sigma_1^2} \int_{\sigma_1^2|t-s|}^{\sigma_1^2(t+s)} \frac{1}{\sqrt{2\pi v}} \exp\left\{-\frac{(r - q)^2}{2v}\right\} dv = \frac{\sigma_\xi^2}{\sigma_1^2} \Gamma_1((t, r), (s, q)).
\end{aligned}$$

The second last equality comes from a convolution of Gaussians.

Having taken care of the covariance, we turn to derive the weak limit. Fix  $N \in \mathbb{N}$ ,  $(t_j, r_j) \in \mathbb{R}_+ \times \mathbb{R}$  and  $\theta_j \in \mathbb{R}$  for  $j = 1, \dots, N$ , and arrange the indices so that  $0 = t_0 \leq t_1 \leq t_2 \leq \dots \leq t_N$ . Then, with  $\ell(k) = i$  iff  $[nt_i] + 1 \leq k \leq [nt_{i+1}]$ ,

$$\begin{aligned}
\sum_{j=1}^N \theta_j \bar{F}_n(t_j, r_j) &= n^{-1/4} \sum_{j=1}^N \theta_j \sum_{k=1}^{[nt_j]} \sum_{x \in \mathbb{Z}} \xi_k(x) p^{[nt_j]-k}([nt_j b] + [r_j \sqrt{n}], x) \\
&= \sum_{\ell=0}^{N-1} \sum_{k=[nt_\ell]+1}^{[nt_{\ell+1}]} n^{-1/4} \sum_{j=\ell+1}^N \theta_j \sum_{x \in \mathbb{Z}} \xi_k(x) p^{[nt_j]-k}([nt_j b] + [r_j \sqrt{n}], x)
\end{aligned}$$

$$= \sum_{k=1}^{\lfloor nt_N \rfloor} n^{-1/4} \sum_{j=\ell(k)+1}^N \theta_j \sum_{x \in \mathbb{Z}} \xi_k(x) p^{\lfloor nt_j \rfloor - k} (\lfloor nt_j b \rfloor + \lfloor r_j \sqrt{n} \rfloor, x) = \sum_{k=1}^{\lfloor nt_N \rfloor} V_{n,k}$$

where the last equality defines variables  $V_{n,k}$ , independent for a fixed  $n$ . We apply the Lindeberg-Feller theorem (Theorem 3.4.5 in [10]) to  $\{V_{n,k}\}$ . (5.11) gives

$$(5.12) \quad \lim_{n \rightarrow \infty} \sum_{k=1}^{\lfloor nt_N \rfloor} \mathbb{E} V_{n,k}^2 = \sum_{i,j=1}^N \theta_i \theta_j \frac{\sigma_\xi^2}{\sigma_1^2} \Gamma_1((t_i, r_i), (t_j, r_j)).$$

As preparation for the second condition of Lindeberg-Feller (the negligibility condition (ii) of Theorem 3.4.5 on p. 129 in [10]) we estimate a moment. Abbreviate  $p_j(x) = p^{\lfloor nt_j \rfloor - k} (\lfloor nt_j b \rfloor + \lfloor r_j \sqrt{n} \rfloor, x)$ . Use  $\mathbb{E}[\xi_n(k)] = 0$ ,  $\mathbb{E}[\xi_t(x)^4] < \infty$  and  $q^k(x, y) \leq Ck^{-1/2}$ .

$$(5.13) \quad \begin{aligned} \mathbb{E}[|V_{n,k}|^4] &\leq \frac{C}{n} \sum_{j=\ell(k)+1}^N \mathbb{E} \left[ \left( \sum_{x \in \mathbb{Z}} \xi_k(x) p_j(x) \right)^4 \right] \\ &\leq \frac{C}{n} \sum_{j=\ell(k)+1}^N \left( \sum_{x,y \in \mathbb{Z}} p_j(x)^2 p_j(y)^2 + \sum_{x \in \mathbb{Z}} p_j(x)^4 \right) \\ &\leq \frac{C}{n} \sum_{j=\ell(k)+1}^N [q^{\lfloor nt_j \rfloor - k}(0, 0)]^2 \leq \frac{C}{n} \sum_{j=\ell(k)+1}^N \frac{1}{(\lfloor nt_j \rfloor - k) \vee 1}. \end{aligned}$$

By Hölder's and Chebyshev's inequalities and by (5.13), for  $\varepsilon > 0$ ,

$$(5.14) \quad \begin{aligned} \sum_{k=1}^{\lfloor nt_N \rfloor} \mathbb{E} [V_{n,k}^2 \mathbf{1}\{|V_{n,k}| \geq \varepsilon\}] &\leq \varepsilon^{-2} \sum_{k=1}^{\lfloor nt_N \rfloor} \mathbb{E}[|V_{n,k}|^4] \leq \frac{C}{n\varepsilon^2} \sum_{j=1}^N \sum_{k=1}^{\lfloor nt_j \rfloor} \frac{1}{(\lfloor nt_j \rfloor - k) \vee 1} \\ &\leq \frac{C}{n\varepsilon^2} (1 + \log n) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

(5.12) and (5.14) verify the hypotheses of the Lindeberg-Feller theorem, which implies

$$\sum_{j=1}^N \theta_j \bar{F}_n(t_j, r_j) = \sum_{k=1}^{\lfloor nt_N \rfloor} V_{n,k} \Rightarrow \sum_{j=1}^N \theta_j F(t_j, r_j).$$

The proof for Proposition 5.2 is complete.  $\square$

We turn to  $\bar{S}_n$ . The hypotheses on the initial increments are now relevant. Let  $\{S(t, r) : t \in \mathbb{R}_+, r \in \mathbb{R}\}$  be the mean-zero Gaussian process with covariance

$$(5.15) \quad E[S(t, r)S(s, q)] = \varrho_0 \Gamma_2((t, r), (s, q)).$$

The function  $\Gamma_2$  has this alternative formulation [33, Chapter 2]:

$$(5.16) \quad \begin{aligned} \Gamma_2((s, q), (t, r)) &= \int_{-\infty}^0 P(B_{\sigma_1^2 s} > q - x) P(B_{\sigma_1^2 t} > r - x) dx \\ &\quad + \int_0^\infty P(B_{\sigma_1^2 s} \leq q - x) P(B_{\sigma_1^2 t} \leq r - x) dx \end{aligned}$$

where  $B_t$  is a standard 1-dimensional Brownian motion.

PROPOSITION 5.3. *Under the conditions in Theorem 3.1, as  $n \rightarrow \infty$ , the finite-dimensional distributions of the process  $\bar{S}_n$  converge weakly to those of  $S$ .*

*Proof.* We have three cases in Theorem 3.1.

Case (a). The proof of the lemma under the i.i.d. and second moment assumption goes via the Lindeberg-Feller theorem and can be found in [38].

Case (b). Now assume that the initial increments  $\eta_0 = \{\eta_0(x)\}_{x \in \mathbb{Z}}$  are a stationary, strongly mixing sequence such that, for some  $\delta > 0$ ,  $\mathbf{E}|\eta_0(0)|^{2+\delta} < \infty$  and the strong mixing coefficients  $\{\alpha(j)\}$  of  $\eta_0$  satisfy  $\sum_{j=0}^{\infty} (j+1)^{2/\delta} \alpha(j) < \infty$ .

We show the distributional convergence

$$(5.17) \quad \sum_{j=1}^N \theta_j \bar{S}_n(t_j, r_j) \Rightarrow \sum_{j=1}^N \theta_j S(t_j, r_j)$$

for a fixed vector of time-space points  $\{(t_j, r_j)\}_{1 \leq j \leq N} \in (\mathbb{R}_+ \times \mathbb{R})^N$  and a fixed real vector  $\bar{\theta} = (\theta_1, \dots, \theta_N)$ . Rewrite the linear combination as

$$(5.18) \quad \sum_{j=1}^N \theta_j \bar{S}_n(t_j, r_j) = \sum_{i \in \mathbb{Z}} a_{n,i} (\eta_0(i) - \mu_0)$$

where

$$(5.19) \quad a_{n,i} = n^{-1/4} \left\{ \mathbf{1}_{\{i>0\}} \sum_{j=1}^N \theta_j P(X_{\lfloor nt_j \rfloor}^{\lfloor nt_j b \rfloor + \lfloor r_j \sqrt{n} \rfloor} \geq i) - \mathbf{1}_{\{i \leq 0\}} \sum_{j=1}^N \theta_j P(X_{\lfloor nt_j \rfloor}^{\lfloor nt_j b \rfloor + \lfloor r_j \sqrt{n} \rfloor} < i) \right\}.$$

Our first task is the  $n \rightarrow \infty$  limit of the variance

$$(5.20) \quad \begin{aligned} \bar{\sigma}_n^2 &= \mathbf{Var} \left[ \sum_{i \in \mathbb{Z}} a_{n,i} (\eta_0(i) - \mu_0) \right] = \sum_{j,k \in \mathbb{Z}} a_{n,j} a_{n,k} \mathbf{Cov}(\eta_0(j), \eta_0(k)) \\ &= \sum_{\ell \in \mathbb{Z}} \mathbf{Cov}(\eta_0(0), \eta_0(\ell)) \sum_{k \in \mathbb{Z}} a_{n,k} a_{n,\ell+k}. \end{aligned}$$

The next lemma is part of Theorem 1.1 in Rio [29]. The quantile function  $Q_X$  of  $|X|$  is defined by

$$Q_X(u) = \inf\{x \in \mathbb{R}_+ : P(|X| > x) \leq u\}, \quad 0 \leq u \leq 1.$$

LEMMA 5.4. *Let  $X$ ,  $Y$ , and  $XY$  be integrable random variables and let  $\alpha = \alpha(\sigma(X), \sigma(Y))$ . Then*

$$(5.21) \quad |\mathbf{Cov}(X, Y)| \leq 4 \int_0^\alpha Q_X(u) Q_Y(u) du.$$



We apply this lemma to show that the series of covariances  $\sum_{k \in \mathbb{Z}} \mathbf{Cov}[\eta_0(0), \eta_0(k)]$  is absolutely convergent. Let  $Q_\eta(u)$  denote the quantile function of  $|\eta_0(0) - \mu_0|$ . Then

$$\begin{aligned} \sum_{\ell \in \mathbb{Z}} |\mathbf{Cov}(\eta_0(0), \eta_0(\ell))| &\leq 4 \sum_{\ell \in \mathbb{Z}} \int_0^{\alpha(|\ell|)} Q_\eta(u)^2 du \leq 4 \int_0^1 \sum_{\ell \in \mathbb{Z}} \mathbf{1}_{\{u \leq \alpha(|\ell|)\}} Q_\eta(u)^2 du \\ &\leq 4 \left[ \int_0^1 \left( \sum_{\ell \in \mathbb{Z}} \mathbf{1}_{\{u \leq \alpha(|\ell|)\}} \right)^{(2+\delta)/\delta} du \right]^{\delta/(2+\delta)} \left[ \int_0^1 Q_\eta(u)^{2+\delta} du \right]^{2/(2+\delta)}. \end{aligned}$$

Since  $\alpha(n) \searrow 0$  as  $n \rightarrow \infty$ , we have

$$\begin{aligned} \int_0^1 \left( \sum_{\ell \in \mathbb{Z}} \mathbf{1}_{\{u \leq \alpha(|\ell|)\}} \right)^{(2+\delta)/\delta} du &= \sum_{j=0}^{\infty} \int_{\alpha(j+1)}^{\alpha(j)} (2j+1)^{(2+\delta)/\delta} du \\ (5.22) \quad &= \sum_{j=0}^{\infty} (2j+1)^{(2+\delta)/\delta} [\alpha(j) - \alpha(j+1)]. \end{aligned}$$

By summation by parts, by  $\alpha(n) \geq 0$ , and by the summability assumption on  $\{\alpha(j)\}$ ,

$$\begin{aligned} &\sum_{j=0}^n (2j+1)^{(2+\delta)/\delta} [\alpha(j) - \alpha(j+1)] \\ &= \alpha(0) - (2n+1)^{(2+\delta)/\delta} \alpha(n+1) + \sum_{j=1}^n \left[ (2j+1)^{(2+\delta)/\delta} - (2j-1)^{(2+\delta)/\delta} \right] \alpha(j) \\ &\leq \alpha(0) + C \sum_{j=1}^n (j+1)^{2/\delta} \alpha(j) \leq \alpha(0) + C \sum_{j=1}^{\infty} (j+1)^{2/\delta} \alpha(j) < \infty. \end{aligned}$$

Consequently the quantity in (5.22) is finite.

For a uniform variable  $U$  on  $(0, 1)$ ,  $Q_\eta(U) \stackrel{d}{=} |\eta_0(0) - \mu_0|$  and so

$$\int_0^1 (Q_\eta(u))^{2+\delta} du = \mathbf{E}[|\eta_0(0) - \mu_0|^{2+\delta}] < \infty.$$

We have shown that

$$\begin{aligned} &\sum_{\ell \in \mathbb{Z}} |\mathbf{Cov}(\eta_0(0), \eta_0(\ell))| \\ (5.23) \quad &\leq C \left( \sum_{j=0}^{\infty} (j+1)^{2/\delta} \alpha(j) \right)^{\delta/(2+\delta)} \left( \mathbf{E}[|\eta_0(0) - \mu_0|^{2+\delta}] \right)^{2/(2+\delta)} < \infty. \end{aligned}$$

Next a lemma for the other series in (5.20).

LEMMA 5.5. *For a finite constant  $C$*

$$(5.24) \quad \sup_{n \in \mathbb{N}, k \in \mathbb{Z}} \left| \sum_{i \in \mathbb{Z}} a_{n,i} a_{n,i+k} \right| \leq C < \infty.$$

For all  $k \in \mathbb{Z}$ ,

$$(5.25) \quad \lim_{n \rightarrow \infty} \sum_{i \in \mathbb{Z}} a_{n,i} a_{n,i+k} = \sum_{1 \leq j_1, j_2 \leq N} \theta_{j_1} \theta_{j_2} \Gamma_2((t_{j_1}, r_{j_1}), (t_{j_2}, r_{j_2})).$$

*Proof.* By the Schwarz inequality

$$\left| \sum_{i \in \mathbb{Z}} a_{n,i} a_{n,i+k} \right| \leq \sum_{i \in \mathbb{Z}} a_{n,i}^2$$

and so (5.24) follows from the finite limit in (5.25) for  $k = 0$ .

To prove (5.25) expand the sum:

$$(5.26) \quad \begin{aligned} & \sum_{i \in \mathbb{Z}} a_{n,i} a_{n,i+k} \\ &= n^{-1/2} \sum_{1 \leq j_1, j_2 \leq N} \theta_{j_1} \theta_{j_2} \sum_{i > 0, i+k > 0} P(X_{\lfloor nt_{j_1} \rfloor}^{\lfloor nt_{j_1} b \rfloor + \lfloor r_{j_1} \sqrt{n} \rfloor} \geq i) P(X_{\lfloor nt_{j_2} \rfloor}^{\lfloor nt_{j_2} b \rfloor + \lfloor r_{j_2} \sqrt{n} \rfloor} \geq i+k) \\ & - n^{-1/2} \sum_{1 \leq j_1, j_2 \leq N} \theta_{j_1} \theta_{j_2} \sum_{i > 0, i+k \leq 0} P(X_{\lfloor nt_{j_1} \rfloor}^{\lfloor nt_{j_1} b \rfloor + \lfloor r_{j_1} \sqrt{n} \rfloor} \geq i) P(X_{\lfloor nt_{j_2} \rfloor}^{\lfloor nt_{j_2} b \rfloor + \lfloor r_{j_2} \sqrt{n} \rfloor} < i+k) \\ & - n^{-1/2} \sum_{1 \leq j_1, j_2 \leq N} \theta_{j_1} \theta_{j_2} \sum_{i \leq 0, i+k > 0} P(X_{\lfloor nt_{j_1} \rfloor}^{\lfloor nt_{j_1} b \rfloor + \lfloor r_{j_1} \sqrt{n} \rfloor} < i) P(X_{\lfloor nt_{j_2} \rfloor}^{\lfloor nt_{j_2} b \rfloor + \lfloor r_{j_2} \sqrt{n} \rfloor} \geq i+k) \\ & + n^{-1/2} \sum_{1 \leq j_1, j_2 \leq N} \theta_{j_1} \theta_{j_2} \sum_{i \leq 0, i+k \leq 0} P(X_{\lfloor nt_{j_1} \rfloor}^{\lfloor nt_{j_1} b \rfloor + \lfloor r_{j_1} \sqrt{n} \rfloor} < i) P(X_{\lfloor nt_{j_2} \rfloor}^{\lfloor nt_{j_2} b \rfloor + \lfloor r_{j_2} \sqrt{n} \rfloor} < i+k). \end{aligned}$$

The middle two terms above are  $O(n^{-1/2})$  and hence vanish as  $n \rightarrow \infty$ .

The individual probabilities converge by the central limit theorem: with  $i = \lfloor x\sqrt{n} \rfloor$ ,

$$\begin{aligned} P(X_{\lfloor nt \rfloor}^{\lfloor nt b \rfloor + \lfloor r \sqrt{n} \rfloor} \geq i) &= P\left(\frac{X_{\lfloor nt \rfloor}^0 + \lfloor nt b \rfloor}{\sqrt{n}} \geq \frac{\lfloor x\sqrt{n} \rfloor - \lfloor r\sqrt{n} \rfloor}{\sqrt{n}}\right) \\ &\rightarrow P(B_{\sigma_1^2 t} \geq x - r) = P(B_{\sigma_1^2 t} \leq r - x). \end{aligned}$$

The first and the fourth sums in (5.26) are handled by Riemann sum arguments, with estimates to control the tails  $|i| \geq C\sqrt{n}$ , to obtain

$$\begin{aligned} & \lim_{n \rightarrow \infty} n^{-1/2} \sum_{i > 0, i+k > 0} P(X_{\lfloor nt_{j_1} \rfloor}^{\lfloor nt_{j_1} b \rfloor + \lfloor r_{j_1} \sqrt{n} \rfloor} \geq i) P(X_{\lfloor nt_{j_2} \rfloor}^{\lfloor nt_{j_2} b \rfloor + \lfloor r_{j_2} \sqrt{n} \rfloor} \geq i+k) \\ &= \int_0^{+\infty} P(B_{\sigma_1^2 t_{j_1}} \leq r_{j_1} - x) P(B_{\sigma_1^2 t_{j_2}} \leq r_{j_2} - x) dx \end{aligned}$$

and

$$\begin{aligned} & \lim_{n \rightarrow \infty} n^{-1/2} \sum_{i \leq 0, i+k \leq 0} P(X_{\lfloor nt_{j_1} \rfloor}^{\lfloor nt_{j_1} b \rfloor + \lfloor r_{j_1} \sqrt{n} \rfloor} < i) P(X_{\lfloor nt_{j_2} \rfloor}^{\lfloor nt_{j_2} b \rfloor + \lfloor r_{j_2} \sqrt{n} \rfloor} < i+k) \\ &= \int_{-\infty}^0 P(B_{\sigma_1^2 t_{j_1}} > r_{j_1} - x) P(B_{\sigma_1^2 t_{j_2}} > r_{j_2} - x) dx. \end{aligned}$$

More details can be found in [38]. By (5.16), this gives the conclusion.  $\square$

We let  $n \rightarrow \infty$  in (5.20), with bounds (5.23) and (5.24) and with limit (5.25) apply the dominated convergence theorem, and recall the definition (3.1) of  $\varrho_0$ , to conclude that

$$(5.27) \quad \lim_{n \rightarrow \infty} \bar{\sigma}_n^2 = \varrho_0 \sum_{1 \leq j_1, j_2 \leq N} \theta_{j_1} \theta_{j_2} \Gamma_2((t_{j_1}, r_{j_1}), (t_{j_2}, r_{j_2})).$$

If the limit on the right of (5.27) vanishes then  $\sum_{j=1}^N \theta_j \bar{S}_n(t_j, r_j)$  converges weakly to zero. We continue by assuming that the limit on the right of (5.27) is strictly positive. The proof of Proposition 5.3 for case (b) will be completed by the following central limit theorem for linear processes due to Peligrad and Utev [26, Theorem 2.2(c)]:

**THEOREM 5.6.** *Let  $\{b_{n,i} : -m_n \leq i \leq m_n, n \in \mathbb{Z}_+\}$  be real numbers that satisfy*

$$(5.28) \quad \limsup_{n \rightarrow \infty} \sum_{i \in \mathbb{Z}} b_{n,i}^2 < \infty$$

and

$$(5.29) \quad \lim_{n \rightarrow \infty} \max_{i \in \mathbb{Z}} |b_{n,i}| = 0,$$

where  $b_{n,i} = 0$  for  $|i| > m_n$ . Let  $\{z(i) : i \in \mathbb{Z}\}$  be a centered, strongly mixing and non-degenerate ( $\text{Var}[z(0)] > 0$ ) stationary sequence with strong mixing coefficients  $\{\alpha(j)\}$  such that

$$(5.30) \quad \text{Var} \left[ \sum_{i=-m_n}^{m_n} b_{n,i} z(i) \right] = 1$$

and there exists  $\delta > 0$  so that  $E|z(0)|^{2+\delta} < \infty$  and  $\sum_{j=0}^{\infty} (j+1)^{2/\delta} \alpha(j) < \infty$ . Then,

$$(5.31) \quad \sum_{i=-m_n}^{m_n} b_{n,i} z(i) \Rightarrow \mathcal{N}(0, 1) \quad \text{as } n \rightarrow \infty.$$

Apply this to  $b_{n,i} = a_{n,i}/\bar{\sigma}_n$  and  $z(i) = \eta_0(i) - \mu_0$ . Since the random walk steps are bounded,  $a_{n,i} = 0$  for large enough  $i$  when  $n$  is given. From (5.25) and (5.27),

$$\lim_{n \rightarrow \infty} \sum_{i \in \mathbb{Z}} b_{n,i}^2 = \varrho_0^{-1} < \infty$$

and condition (5.28) is satisfied. By (5.19)  $|a_{n,i}| \leq n^{-1/4} \sum_{j=1}^N |\theta_j|$  and thereby

$$\max_{i \in \mathbb{Z}} |b_{n,i}| \leq \frac{1}{n^{1/4} \bar{\sigma}_n} \sum_{j=1}^N |\theta_j| \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

verifying (5.29). Theorem 5.6 gives  $\bar{\sigma}_n^{-1} \sum_{i \in \mathbb{Z}} a_{n,i} (\eta_0(i) - \mu_0) \Rightarrow \mathcal{N}(0, 1)$  which, combined with (5.27), yields

$$\sum_{j=1}^N \theta_j \bar{S}_n(t_j, r_j) \Rightarrow \mathcal{N} \left( 0, \varrho_0 \sum_{1 \leq j_1, j_2 \leq N} \theta_{j_1} \theta_{j_2} \Gamma_2((t_{j_1}, r_{j_1}), (t_{j_2}, r_{j_2})) \right).$$

This concludes the proof of Proposition 5.3 of case (b).

Case (c). It remains to consider the case where the initial increment sequence  $\eta_0$  is defined by (3.10). Let  $\ell(n)$  be any increasing sequence such that  $\lim_{n \rightarrow \infty} n/\sqrt{\ell(n)} = 0$ . Split (5.18), with  $\mu_0 = 0$  now, into two terms:

$$(5.32) \quad \begin{aligned} \sum_{j=1}^N \theta_j \bar{S}_n(t_j, r_j) &= \sum_{i \in \mathbb{Z}} a_{n,i} \eta_0(i) = \sum_{k=0}^{\ell(n)-1} \sum_{i,j \in \mathbb{Z}} a_{n,i} \xi_{-k}(j) [p^k(i, j) - p^k(i-1, j)] \\ &+ \sum_{k=\ell(n)}^{\infty} \sum_{i,j \in \mathbb{Z}} a_{n,i} \xi_{-k}(j) [p^k(i, j) - p^k(i-1, j)] = T_{n,1} + T_{n,2}. \end{aligned}$$

In the next lemma we show that the term  $T_{n,2}$  is negligible.

LEMMA 5.7. *With  $\lim_{n \rightarrow \infty} n/\sqrt{\ell(n)} = 0$ , we have  $\lim_{n \rightarrow \infty} \mathbb{E}[|T_{n,2}|^2] \rightarrow 0$ .*

*Proof.* Due to the bounded range of the jump kernel  $p(0, j)$ , after the first equality sign below the sums over  $i_1$  and  $i_2$  have  $O(n)$  terms, and for a fixed  $k$  the sum over  $j$  is also finite.

$$(5.33) \quad \begin{aligned} \mathbb{E}[|T_{n,2}|^2] &= \sigma_\xi^2 \sum_{k=\ell(n)}^{\infty} \sum_{i_1, i_2 \in \mathbb{Z}} a_{n,i_1} a_{n,i_2} \sum_{j \in \mathbb{Z}} (p_{i_1,j}^k - p_{i_1-1,j}^k)(p_{i_2,j}^k - p_{i_2-1,j}^k) \\ &= \sigma_\xi^2 \sum_{k=\ell(n)}^{\infty} \sum_{i_1, i_2 \in \mathbb{Z}} a_{n,i_1} a_{n,i_2} (2q_{i_2-i_1,0}^k - q_{i_2-i_1+1,0}^k - q_{i_2-i_1-1,0}^k) \\ &\leq \sigma_\xi^2 \sum_{j \in \mathbb{Z}} \left| \sum_{k=\ell(n)}^{\infty} (2q_{j,0}^k - q_{j+1,0}^k - q_{j-1,0}^k) \right| \cdot \left| \sum_{i \in \mathbb{Z}} a_{n,i} a_{n,i+j} \right|. \end{aligned}$$

Bound the middle sum using the characteristic function  $\phi_q(\theta) = \sum_{j \in \mathbb{Z}} q(0, j) e^{ij\theta}$ .

$$\begin{aligned} &\left| \sum_{k=\ell(n)}^{\infty} (2q_{j,0}^k - q_{j+1,0}^k - q_{j-1,0}^k) \right| \\ &= \left| \sum_{k=\ell(n)}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi_q^k(\theta) (2e^{-\iota j\theta} - e^{-\iota(j+1)\theta} - e^{-\iota(j-1)\theta}) d\theta \right| \\ &= \left| \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\phi_q^{\ell(n)}(\theta)(1 - \cos \theta)}{1 - \phi_q(\theta)} e^{-\iota j\theta} d\theta \right| \leq C \int_{-\pi}^{\pi} \phi_q^{\ell(n)}(\theta) d\theta = C q^{\ell(n)}(0, 0) \\ &\leq \frac{C}{\sqrt{\ell(n)}}. \end{aligned}$$

The ratio  $(1 - \cos \theta)/(1 - \phi_q(\theta))$  is bounded over  $[-\pi, \pi]$  because it has a finite limit at  $\theta = 0$  and by the span 1 property  $\phi_q(\theta) = 1$  only at  $\theta = 0$ .

The sum over  $j$  on line (5.33) has  $O(n)$  nonzero terms. Recalling (5.24), we conclude that  $\mathbb{E}[|T_{n,2}|^2] \leq Cn/\sqrt{\ell(n)} \rightarrow 0$  as  $n \rightarrow \infty$ .  $\square$

Let  $U_{n,k}(j) = \xi_{-k}(j) \sum_{i \in \mathbb{Z}} a_{n,i} (p_{i,j}^k - p_{i-1,j}^k)$  so that  $T_{n,1} = \sum_{k=0}^{\ell(n)-1} \sum_{j \in \mathbb{Z}} U_{n,k}(j)$  is a finite sum of independent mean zero random variables. To prove the goal (5.17), the

Lindeberg-Feller theorem shows that  $T_{n,1} \Rightarrow \sum_{j=1}^N \theta_j S(t_j, r_j)$ . First the limiting variance.

$$\begin{aligned}
 \mathbb{E}[T_{n,1}^2] &= \sigma_\xi^2 \sum_{j \in \mathbb{Z}} \sum_{k=0}^{\ell(n)-1} [2q^k(j, 0) - q^k(j+1, 0) - q^k(j-1, 0)] \sum_{i \in \mathbb{Z}} a_{n,i} a_{n,i+j} \\
 (5.34) \quad &= \sigma_\xi^2 \sum_{j \in \mathbb{Z}} [a(j-1) + a(j+1) - 2a(j)] \sum_{i \in \mathbb{Z}} a_{n,i} a_{n,i+j} - \mathbb{E}[T_{n,2}^2] \\
 &\xrightarrow{n \rightarrow \infty} \frac{\sigma_\xi^2}{\sigma_1^2} \sum_{1 \leq j_1, j_2 \leq N} \theta_{j_1} \theta_{j_2} \Gamma_2((t_{j_1}, r_{j_1}), (t_{j_2}, r_{j_2})).
 \end{aligned}$$

The last limit came from a combination of (2.23) and Lemmas 5.5 and 5.7.

For the Lindeberg condition (condition (ii) of Theorem 3.4.5 on p. 129 in [10]) the task is to show that, for all  $\varepsilon > 0$ ,

$$(5.35) \quad \sum_{j \in \mathbb{Z}} \sum_{k=0}^{\ell(n)-1} \mathbb{E}[U_{n,k}^2(j) \mathbf{1}\{|U_{n,k}(j)| \geq \varepsilon\}] \rightarrow 0.$$

By Schwarz and Markov inequalities,

$$\begin{aligned}
 \sum_{j \in \mathbb{Z}} \sum_{k=0}^{\ell(n)-1} \mathbb{E}[U_{n,k}^2(j) \mathbf{1}\{|U_{n,k}(j)| \geq \varepsilon\}] &\leq \frac{1}{\varepsilon^2} \sum_{j \in \mathbb{Z}} \sum_{k=0}^{\ell(n)-1} \mathbb{E}[U_{n,k}^4(j)] \\
 (5.36) \quad &= \frac{\mathbb{E}[\xi_0^4(0)]}{\varepsilon^2} \sum_{j \in \mathbb{Z}} \sum_{k=0}^{\ell(n)-1} \left( \sum_{i \in \mathbb{Z}} a_{n,i} [p^k(i, j) - p^k(i-1, j)] \right)^4.
 \end{aligned}$$

To two powers of the quantity in parentheses apply the inequality

$$\sum_{i \in \mathbb{Z}} |a_{n,i}| [p^k(i, j) + p^k(i-1, j)] \leq Cn^{-1/4}.$$

Then line (5.36) is bounded above by

$$(5.37) \quad \frac{C}{\varepsilon^2 n^{1/2}} \sum_{j \in \mathbb{Z}} \sum_{k=0}^{\ell(n)-1} \left( \sum_{i \in \mathbb{Z}} a_{n,i} [p^k(i, j) - p^k(i-1, j)] \right)^2$$

which vanishes as  $n \rightarrow \infty$  because the finite limit of the double sum of squares is exactly what was treated in (5.34). This verifies (5.35). Collecting the pieces, we have showed that  $T_{n,1} \Rightarrow \sum_{j=1}^N \theta_j S(t_j, r_j)$ . This, together with (5.32) and Lemma 5.7, implies the goal (5.17). Proposition 5.3 has now been proved also for case (c) of Theorem 3.1.  $\square$

To complete the proof of Theorem 3.1, apply Propositions 5.2 and 5.3 to the right-hand side of decomposition (5.3), and use the independence of the processes  $\bar{S}_n$  and  $\bar{F}_n$ .

## 6. HEIGHT FLUCTUATIONS: PROCESS-LEVEL CONVERGENCE

In this section we prove Theorem 3.4. To simplify the exposition we take  $Q = [0, 1]^2$ . Theorem 2 in Bickel and Wichura [3] gives the following necessary and sufficient condition for weak convergence  $X_n \Rightarrow X$  of  $D_2$ -valued processes.

- (i) (Convergence of finite-dimensional distributions.) For all finite sets  $\{(t_i, r_i)\}_{i=1}^N \subset [0, 1]^2$ , we have

$$(X_n(t_1, r_1), \dots, X_n(t_N, r_N)) \Rightarrow (X(t_1, r_1), \dots, X(t_N, r_N)).$$

- (ii) (Tightness.)  $\forall \varepsilon > 0$ ,  $\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} P\{w'_\delta(X_n) \geq \varepsilon\} = 0$ , for the modulus

$$w'_\delta(x) = \inf_{\Delta} \max_{G \in \Delta} \sup_{(t,r),(s,q) \in G} |x(t, r) - x(s, q)|, \quad x \in D_2,$$

where the infimum is over partitions  $\Delta$  of  $[0, 1]^2$  formed by finitely many lines parallel to the coordinate axes and such that any element  $G$  of  $\Delta$  is a left-closed, right-open rectangle with diameter at least  $\delta$ .

We proved the finite-dimensional marginal convergence (i) in Theorem 3.1. For the tightness proof that follows we check the sufficient conditions given by the next lemma (Proposition 2 in [21]) in terms of the modulus of continuity

$$(6.1) \quad w_\delta(x) = \sup_{\substack{(t,r),(s,q) \in [0,1]^2 \\ |(t,r)-(s,q)| < \delta}} |x(t, r) - x(s, q)|, \quad x \in D_2.$$

LEMMA 6.1. *Let  $\{X_n\}$  be a sequence of  $D_2$ -valued processes. Assume that there exists a decreasing sequence  $\delta_n \searrow 0$  such that*

- (i)  $\exists \beta > 0$ ,  $\kappa > 2$ , and  $C > 0$  such that, for all large enough  $n$ ,

$$(6.2) \quad E(|X_n(t, r) - X_n(s, q)|^\beta) \leq C|(t, r) - (s, q)|^\kappa$$

holds for all  $(t, r), (s, q) \in [0, 1]^2$  at distance  $|(t, r) - (s, q)| > \delta_n$ ;

- (ii)  $\forall \varepsilon, \eta > 0$ , there exists an  $n_0 > 0$  such that for all  $n \geq n_0$ ,

$$(6.3) \quad P\{w_{\delta_n}(X_n) > \varepsilon\} < \eta.$$

Then, for all  $\varepsilon, \eta > 0$ , there exist  $0 < \delta < 1$  and integer  $n_0 < \infty$  such that

$$P\{w_\delta(X_n) \geq \varepsilon\} \leq \eta \quad \forall n \geq n_0.$$

The two tightness conditions (i) and (ii) are checked in Lemmas 6.2 and 6.5.

LEMMA 6.2. *Assume the assumptions of Theorem 3.4. Fix  $\kappa$  and  $\gamma$  such that  $2 < \kappa < 3$  and  $0 < \gamma \leq \frac{3}{\kappa}$  and let  $\delta_n = n^{-\gamma}$ . Then, there exists a constant  $C > 0$  such that for all sufficiently large  $n$ ,*

$$(6.4) \quad \mathbf{E}(|\bar{h}_n(t, r) - \bar{h}_n(s, q)|^{12}) \leq C|(t, r) - (s, q)|^\kappa$$

for all  $t, s, r, q \in [0, 1]$  with  $|(t, r) - (s, q)| > n^{-\gamma}$ .

*Proof.* From decomposition (5.3), for a constant  $C$ ,

$$(6.5) \quad \begin{aligned} C^{-1} \mathbf{E}(|\bar{h}_n(t, r) - \bar{h}_n(s, q)|^{12}) &\leq |\bar{H}_n(t, r) - \bar{H}_n(s, q)|^{12} \\ &+ \mathbf{E}(|\bar{F}_n(t, r) - \bar{F}_n(s, q)|^{12}) + \mathbf{E}(|\bar{S}_n(t, r) - \bar{S}_n(s, q)|^{12}). \end{aligned}$$

Estimate (6.4) comes by treating each term on the right-hand side of (6.5) in turn.

As observed in the early part of Section 5,  $\overline{H}_n(t, r) = O(n^{-1/4})$  uniformly, and the first term on the right of (6.5) satisfies

$$(6.6) \quad |\overline{H}_n(t, r) - \overline{H}_n(s, q)|^{12} \leq Cn^{-3} \leq Cn^{-\gamma\kappa} \leq C|(t, r) - (s, q)|^\kappa.$$

The following argument will be used more than once: if  $\{\zeta_i\}$  are i.i.d. mean zero variables with a finite 12th moment and  $|a_i| \vee 1 \leq B$ , then

$$(6.7) \quad \begin{aligned} E \left[ \left( \sum_i a_i \zeta_i \right)^{12} \right] &= E \sum_{i_1, i_2, \dots, i_{12}} a_{i_1} \cdots a_{i_{12}} \zeta_{i_1} \cdots \zeta_{i_{12}} \\ &\leq C \sum_{\substack{1 \leq k \leq 6 \\ m_i \geq 2: m_1 + \dots + m_k = 12}} \left( \sum_{i_1} |a_{i_1}|^{m_1} \right) \left( \sum_{i_2} |a_{i_2}|^{m_2} \right) \cdots \left( \sum_{i_k} |a_{i_k}|^{m_k} \right) \\ &\leq B^{10} C \left( 1 + \sum_i a_i^2 \right)^6 \end{aligned}$$

where  $C$  is a combinatorial constant.

For the second term on the right of (6.5), recalling (5.5), and with  $t \geq s$ ,

$$\begin{aligned} &\overline{F}_n(t, r) - \overline{F}_n(s, q) \\ &= n^{-1/4} \sum_{k=1}^{\lfloor ns \rfloor} \sum_{x \in \mathbb{Z}} \xi_k(x) \left[ P(X_{\lfloor nt \rfloor - k}^{\lfloor nt b \rfloor + \lfloor r \sqrt{n} \rfloor} = x) - P(X_{\lfloor ns \rfloor - k}^{\lfloor ns b \rfloor + \lfloor q \sqrt{n} \rfloor} = x) \right] \\ &\quad + n^{-1/4} \sum_{k=\lfloor ns \rfloor + 1}^{\lfloor nt \rfloor} \sum_{x \in \mathbb{Z}} \xi_k(x) P(X_{\lfloor nt \rfloor - k}^{\lfloor nt b \rfloor + \lfloor r \sqrt{n} \rfloor} = x). \end{aligned}$$

Apply (6.7):

$$\begin{aligned} &\mathbb{E} |\overline{F}_n(t, r) - \overline{F}_n(s, q)|^{12} \\ &\leq Cn^{-3} \left\{ 1 + \sum_{k=1}^{\lfloor ns \rfloor} \sum_{x \in \mathbb{Z}} \left[ P(X_{\lfloor nt \rfloor - k}^{\lfloor nt b \rfloor + \lfloor r \sqrt{n} \rfloor} = x) - P(X_{\lfloor ns \rfloor - k}^{\lfloor ns b \rfloor + \lfloor q \sqrt{n} \rfloor} = x) \right]^2 \right. \\ &\quad \left. + \sum_{k=\lfloor ns \rfloor + 1}^{\lfloor nt \rfloor} \sum_{x \in \mathbb{Z}} P(X_{\lfloor nt \rfloor - k}^{\lfloor nt b \rfloor + \lfloor r \sqrt{n} \rfloor} = x)^2 \right\}^6. \end{aligned}$$



The sums inside the braces develop as follows.

$$\begin{aligned}
(6.8) \quad & \sum_{k=1}^{\lfloor ns \rfloor} \sum_{x \in \mathbb{Z}} \left[ P(X_{\lfloor nt \rfloor - k}^{\lfloor nt b \rfloor + \lfloor r \sqrt{n} \rfloor} = x) - P(X_{\lfloor ns \rfloor - k}^{\lfloor ns b \rfloor + \lfloor q \sqrt{n} \rfloor} = x) \right]^2 \\
& + \sum_{k=\lfloor ns \rfloor + 1}^{\lfloor nt \rfloor} \sum_{x \in \mathbb{Z}} P(X_{\lfloor nt \rfloor - k}^{\lfloor nt b \rfloor + \lfloor r \sqrt{n} \rfloor} = x)^2 \\
& = \sum_{k=1}^{\lfloor nt \rfloor} P(X_{\lfloor nt \rfloor - k}^{\lfloor nt b \rfloor + \lfloor r \sqrt{n} \rfloor} - \tilde{X}_{\lfloor nt \rfloor - k}^{\lfloor nt b \rfloor + \lfloor r \sqrt{n} \rfloor} = 0) \\
& + \sum_{k=1}^{\lfloor ns \rfloor} P(X_{\lfloor ns \rfloor - k}^{\lfloor ns b \rfloor + \lfloor q \sqrt{n} \rfloor} - \tilde{X}_{\lfloor ns \rfloor - k}^{\lfloor ns b \rfloor + \lfloor q \sqrt{n} \rfloor} = 0) \\
& - 2 \sum_{k=1}^{\lfloor ns \rfloor} P(X_{\lfloor nt \rfloor - k}^{\lfloor nt b \rfloor + \lfloor r \sqrt{n} \rfloor} - \tilde{X}_{\lfloor ns \rfloor - k}^{\lfloor ns b \rfloor + \lfloor q \sqrt{n} \rfloor} = 0),
\end{aligned}$$

where  $X_\cdot$  and  $\tilde{X}_\cdot$  are independent random walks with transition probability  $p$ .

Recalling transition probability  $q$  from (2.11),

$$\sum_{k=1}^{\lfloor nt \rfloor} P(X_{\lfloor nt \rfloor - k}^{\lfloor nt b \rfloor + \lfloor r \sqrt{n} \rfloor} - \tilde{X}_{\lfloor nt \rfloor - k}^{\lfloor nt b \rfloor + \lfloor r \sqrt{n} \rfloor} = 0) = \sum_{k=1}^{\lfloor nt \rfloor} q^{\lfloor nt \rfloor - k}(0, 0) = \sum_{k=0}^{\lfloor nt \rfloor - 1} q^k(0, 0).$$

Similarly,

$$\sum_{k=1}^{\lfloor ns \rfloor} P(X_{\lfloor ns \rfloor - k}^{\lfloor ns b \rfloor + \lfloor q \sqrt{n} \rfloor} - \tilde{X}_{\lfloor ns \rfloor - k}^{\lfloor ns b \rfloor + \lfloor q \sqrt{n} \rfloor} = 0) = \sum_{k=0}^{\lfloor ns \rfloor - 1} q^k(0, 0).$$

And,

$$\begin{aligned}
& \sum_{k=1}^{\lfloor ns \rfloor} P(X_{\lfloor nt \rfloor - k}^{\lfloor nt b \rfloor + \lfloor r \sqrt{n} \rfloor} - \tilde{X}_{\lfloor ns \rfloor - k}^{\lfloor ns b \rfloor + \lfloor q \sqrt{n} \rfloor} = 0) = \sum_{k=0}^{\lfloor ns \rfloor - 1} P(X_{\lfloor nt \rfloor - \lfloor ns \rfloor + k}^{\lfloor nt b \rfloor + \lfloor r \sqrt{n} \rfloor} - \tilde{X}_k^{\lfloor ns b \rfloor + \lfloor q \sqrt{n} \rfloor} = 0) \\
& = \sum_{k=0}^{\lfloor ns \rfloor - 1} E \left[ P(X_k^{\lfloor nt b \rfloor + \lfloor r \sqrt{n} \rfloor} - \tilde{X}_k^{\lfloor ns b \rfloor + \lfloor q \sqrt{n} \rfloor} = 0) \right] \\
& = E \left[ \sum_{k=0}^{\lfloor ns \rfloor - 1} q^k (X_{\lfloor nt \rfloor - \lfloor ns \rfloor}^{\lfloor nt b \rfloor + \lfloor r \sqrt{n} \rfloor} - \lfloor ns b \rfloor - \lfloor q \sqrt{n} \rfloor, 0) \right].
\end{aligned}$$

In sum, we rewrite the right-hand side of (6.8) as

$$(6.9) \quad \sum_{k=0}^{\lfloor nt \rfloor - 1} q^k(0,0) + \sum_{k=0}^{\lfloor ns \rfloor - 1} q^k(0,0) - 2E \left\{ \sum_{k=0}^{\lfloor ns \rfloor - 1} q^k(X_{\lfloor nt \rfloor - \lfloor ns \rfloor}^{\lfloor nt b \rfloor + \lfloor r \sqrt{n} \rfloor} - \lfloor ns b \rfloor - \lfloor q \sqrt{n} \rfloor, 0) \right\} \\ = \sum_{k=\lfloor ns \rfloor}^{\lfloor nt \rfloor - 1} q^k(0,0) + 2E \left\{ \sum_{k=0}^{\lfloor ns \rfloor - 1} \left[ q^k(0,0) - q^k(X_{\lfloor nt \rfloor - \lfloor ns \rfloor}^{\lfloor nt b \rfloor + \lfloor r \sqrt{n} \rfloor} - \lfloor ns b \rfloor - \lfloor q \sqrt{n} \rfloor, 0) \right] \right\}.$$

The first term in (6.9) is bounded by

$$(6.10) \quad \sum_{k=\lfloor ns \rfloor}^{\lfloor nt \rfloor - 1} q^k(0,0) \leq \sum_{k=\lfloor ns \rfloor}^{\lfloor nt \rfloor - 1} \frac{C}{\sqrt{k}} \leq \sum_{k=1}^{\lfloor nt \rfloor - \lfloor ns \rfloor} \frac{C}{\sqrt{k}} \leq C \left[ 1 + \sqrt{(t-s)n} \right].$$

To bound the second term in (6.9), observe first that the terms in the potential kernel (2.12) are nonnegative:

$$q^k(x,0) = \sum_{y \in \mathbb{Z}} p^k(x,y) p^k(0,y) \leq \frac{1}{2} \sum_{y \in \mathbb{Z}} \left[ (p^k(x,y))^2 + (p^k(0,y))^2 \right] = \sum_{y \in \mathbb{Z}} (p^k(0,y))^2 \\ = q^k(0,0).$$

Now the second term in (6.9) is bounded by

$$(6.11) \quad E \left\{ \sum_{k=0}^{\lfloor ns \rfloor - 1} \left[ q^k(0,0) - q^k(X_{\lfloor nt \rfloor - \lfloor ns \rfloor}^{\lfloor nt b \rfloor + \lfloor r \sqrt{n} \rfloor} - \lfloor ns b \rfloor - \lfloor q \sqrt{n} \rfloor, 0) \right] \right\} \\ \leq E \left[ a(X_{\lfloor nt \rfloor - \lfloor ns \rfloor}^{\lfloor nt b \rfloor + \lfloor r \sqrt{n} \rfloor} - \lfloor ns b \rfloor - \lfloor q \sqrt{n} \rfloor) \right] \\ \leq CE |X_{\lfloor nt \rfloor - \lfloor ns \rfloor}^{\lfloor nt b \rfloor + \lfloor r \sqrt{n} \rfloor} - \lfloor ns b \rfloor - \lfloor q \sqrt{n} \rfloor| \\ \leq C(|t-s|^{1/2} + |r-q|) \sqrt{n} + C,$$

where the second inequality is due to

$$(6.12) \quad \lim_{x \rightarrow \pm\infty} |x|^{-1} a(x) = \frac{1}{2} \sigma_1^{-2},$$

from P28.4 on p. 345 of [34] and by the symmetry of  $a(x)$ .

Combining (6.10) and (6.11) gives a bound for (6.8), and hence the second term on the right of (6.5) satisfies

$$(6.13) \quad \mathbb{E} (|\overline{F}_n(t,r) - \overline{F}_n(s,q)|^{12}) \leq Cn^{-3} \left[ \left( \sqrt{|t-s|} + |r-q| \right) \sqrt{n} + 1 \right]^6.$$

For the third term on the right of (6.5), from (5.6),

$$\overline{S}_n(t,r) - \overline{S}_n(s,q) \\ = n^{-1/4} \sum_{i>0} (\eta_0(i) - \mu_0) \left[ P(i \leq X_{\lfloor nt \rfloor}^{\lfloor nt b \rfloor + \lfloor r \sqrt{n} \rfloor}) - P(i \leq X_{\lfloor ns \rfloor}^{\lfloor ns b \rfloor + \lfloor q \sqrt{n} \rfloor}) \right] \\ - n^{-1/4} \sum_{i \leq 0} (\eta_0(i) - \mu_0) \left[ P(i > X_{\lfloor nt \rfloor}^{\lfloor nt b \rfloor + \lfloor r \sqrt{n} \rfloor}) - P(i > X_{\lfloor ns \rfloor}^{\lfloor ns b \rfloor + \lfloor q \sqrt{n} \rfloor}) \right].$$

Note that  $X_t^k \stackrel{d}{=} k + X_t^0$  and define events

$$\begin{aligned} A_{1,i} &= \left\{ X_{[nt]}^i \geq -[ntb] - \lfloor r\sqrt{n} \rfloor, X_{[ns]}^i < -[nsb] - \lfloor q\sqrt{n} \rfloor \right\}, \\ A_{2,i} &= \left\{ X_{[nt]}^i < -[ntb] - \lfloor r\sqrt{n} \rfloor, X_{[ns]}^i \geq -[nsb] - \lfloor q\sqrt{n} \rfloor \right\}. \end{aligned}$$

Then we can rewrite the difference above as

$$(6.14) \quad \bar{S}_n(t, r) - \bar{S}_n(s, q) = n^{-1/4} \sum_{i \in \mathbb{Z}} (\eta_0(-i) - \mu_0) [P(A_{1,i}) - P(A_{2,i})].$$

We prove an intermediate bound where the different cases (a), (b) and (c) of Theorem 3.4 are felt.

LEMMA 6.3. *Suppose the initial increment sequence  $\{\eta_0(x)\}$  satisfies one of the assumptions (a), (b) and (c) of Theorem 3.4. Then we have the inequality*

$$(6.15) \quad \mathbf{E}[(\bar{S}_n(t, r) - \bar{S}_n(s, q))^2] \leq Cn^{-3} \left\{ 1 + \sum_{m \in \mathbb{Z}} [P(A_{1,m}) + P(A_{2,m})] \right\}^6.$$

*Proof.* Case (a). The case of i.i.d.  $\{\eta_0(x)\}$  is handled by the argument in (6.7), beginning with (6.14). Details can be found in [38].

Case (b). For the case of strongly mixing initial increments we state a lemma from Rio's lectures (see Theorem 2.2 and the derivation of equation (C.6) in [29]).

LEMMA 6.4. (a) *Let  $m \in \mathbb{N}$ ,  $\{X_i\}_{i \in \mathbb{N}}$  centered real random variables with strong mixing coefficients  $\{\alpha(k)\}_{k \geq 0}$  and define*

$$\alpha^{-1}(u) = \inf\{k \in \mathbb{Z}_+ : \alpha(k) \leq u\} = \sum_{i \geq 0} \mathbf{1}_{\{u < \alpha(i)\}}.$$

*Assume  $E|X_i|^{2m} < \infty$  and let  $Q_k(u)$  be the quantile function of  $|X_k|$ . Let  $S_n = \sum_{k=1}^n X_k$ . Then there exist positive constants  $a_m$  and  $b_m$  such that*

$$(6.16) \quad \begin{aligned} E(S_n^{2m}) &\leq a_m \left( \int_0^1 \sum_{k=1}^n [\alpha^{-1}(u) \wedge n] Q_k^2(u) du \right)^m \\ &\quad + b_m \sum_{k=1}^n \int_0^1 [\alpha^{-1}(u) \wedge n]^{2m-1} Q_k^{2m}(u) du. \end{aligned}$$

(b) *Suppose centered  $\{X_i\}_{i \in \mathbb{N}}$  have finite  $r$ th moment. Then for  $p \in [1, r)$  there exists a constant  $c_p > 0$  such that*

$$(6.17) \quad \begin{aligned} &\sum_{k=1}^n \int_0^1 [\alpha^{-1}(u) \wedge n]^{p-1} Q_k^p(u) du \\ &\leq c_p \left( \sum_{i=0}^n (i+1)^{(pr-2r+p)/(r-p)} \alpha(i) \right)^{1-p/r} \sum_{k=1}^n (E|X_k|^r)^{p/r}. \end{aligned}$$

Apply this lemma to representation (6.14). Note that  $k_n = \#\{i \in \mathbb{Z} : P(A_{1,i}) - P(A_{2,i}) \neq 0\} = O(n)$  due to the bounded support of  $p$  (2.1). Letting  $Q_i$  be the quantile function of  $|\eta_0(-i) - \mu_0| \cdot |P(A_{1,i}) - P(A_{2,i})|$  and  $m = 6$ , (6.16) gives

$$\begin{aligned} & \mathbf{E}[(\bar{S}_n(t, r) - \bar{S}_n(s, q))^{12}] \\ & \leq Cn^{-3} \left[ \left( \sum_{i \in \mathbb{Z}} \int_0^1 [\alpha^{-1}(u) \wedge k_n] Q_i^2(u) du \right)^6 + \sum_{j \in \mathbb{Z}} \int_0^1 [\alpha^{-1}(u) \wedge k_n]^{11} Q_j^{12}(u) du \right]. \end{aligned}$$

Let  $p = 2, r = 12$  in (6.17), we can get an upper bound for  $\sum_{i \in \mathbb{Z}} \int_0^1 [\alpha^{-1}(u) \wedge k_n] Q_i^2(u) du$ ,

$$\begin{aligned} & \sum_{i \in \mathbb{Z}} \int_0^1 [\alpha^{-1}(u) \wedge k_n] Q_i^2(u) du \\ & \leq c_2 \left( \sum_{i=0}^{k_n} (i+1)^{1/5} \alpha(i) \right)^{5/6} \sum_{j \in \mathbb{Z}} \left( \mathbf{E} |\eta_0(-j) - \mu_0|^{12} \right)^{1/6} |P(A_{1,j}) - P(A_{2,j})|^2 \\ & \leq C \left( \sum_{i=0}^{k_n} (i+1)^{1/5} \alpha(i) \right)^{5/6} \sum_{j \in \mathbb{Z}} [P(A_{1,j}) + P(A_{2,j})] \leq C \sum_{j \in \mathbb{Z}} [P(A_{1,j}) + P(A_{2,j})]. \end{aligned}$$

By the same token, let  $p = 12, r = 12 + \delta$  in (6.17), we can show that

$$\begin{aligned} & \sum_{j \in \mathbb{Z}} \int_0^1 [\alpha^{-1}(u) \wedge k_n]^{11} Q_j^{12}(u) du \\ & \leq C \left( \sum_{i=0}^{k_n} (i+1)^{10+132/\delta} \alpha(i) \right)^{\delta/(12+\delta)} \sum_{j \in \mathbb{Z}} [P(A_{1,j}) + P(A_{2,j})] \\ & \leq C \sum_{j \in \mathbb{Z}} [P(A_{1,j}) + P(A_{2,j})]. \end{aligned}$$

Hence,

$$\begin{aligned} & \mathbf{E}[(\bar{S}_n(t, r) - \bar{S}_n(s, q))^{12}] \\ (6.18) \quad & \leq Cn^{-3} \left\{ \left( \sum_{j \in \mathbb{Z}} [P(A_{1,j}) + P(A_{2,j})] \right)^6 + \sum_{j \in \mathbb{Z}} [P(A_{1,j}) + P(A_{2,j})] \right\} \\ & \leq Cn^{-3} \left\{ 1 + \sum_{m \in \mathbb{Z}} [P(A_{1,m}) + P(A_{2,m})] \right\}^6. \end{aligned}$$

Case (c). From (6.14) and (3.10),

$$\bar{S}_n(t, r) - \bar{S}_n(s, q) = n^{-1/4} \sum_{i, j \in \mathbb{Z}} \sum_{k=0}^{\infty} \xi_{-k}(j) (p_{0,j+i}^k - p_{0,j+i+1}^k) [P(A_{1,i}) - P(A_{2,i})].$$

Then by (6.7)

$$(6.19) \quad \begin{aligned} & \mathbf{E}[(\overline{S}_n(t, r) - \overline{S}_n(s, q))^{12}] \\ & \leq Cn^{-3} \left( 1 + \sum_{j \in \mathbb{Z}} \sum_{k=0}^{\infty} \left| \sum_{i \in \mathbb{Z}} (p_{0,j+i}^k - p_{0,j+i+1}^k) [P(A_{1,i}) - P(A_{2,i})] \right|^2 \right)^6 \end{aligned}$$

Expand the sum of squares inside the parentheses:

$$\begin{aligned} & \sum_{j \in \mathbb{Z}} \sum_{k=0}^{\infty} \sum_{i_1, i_2 \in \mathbb{Z}} (p_{0,j+i_1}^k - p_{0,j+i_1+1}^k) (p_{0,j+i_2}^k - p_{0,j+i_2+1}^k) \\ & \quad \times [P(A_{1,i_1}) - P(A_{2,i_1})] [P(A_{1,i_2}) - P(A_{2,i_2})] \\ & = \sum_{i, \ell \in \mathbb{Z}} [a(\ell-1) + a(\ell+1) - 2a(\ell)] \cdot [P(A_{1,i}) - P(A_{2,i})] [P(A_{1,i+\ell}) - P(A_{2,i+\ell})] \\ & \leq \sum_{\ell \in \mathbb{Z}} [a(\ell-1) + a(\ell+1) - 2a(\ell)] \\ & \quad \times \sum_{i \in \mathbb{Z}} \frac{1}{2} \left\{ [P(A_{1,i}) - P(A_{2,i})]^2 + [P(A_{1,i+\ell}) - P(A_{2,i+\ell})]^2 \right\} \\ & = \frac{1}{\sigma_1^2} \sum_{i \in \mathbb{Z}} [P(A_{1,i}) - P(A_{2,i})]^2 \leq \frac{1}{\sigma_1^2} \sum_{i \in \mathbb{Z}} [P(A_{1,i}) + P(A_{2,i})]. \end{aligned}$$

where the last equality is from (2.23). Inserting this into (6.19) gives

$$\mathbf{E}[(\overline{S}_n(t, r) - \overline{S}_n(s, q))^{12}] \leq Cn^{-3} \left\{ 1 + \sum_{i \in \mathbb{Z}} [P(A_{1,i}) + P(A_{2,i})] \right\}^6.$$

The proof of Lemma 6.3 is complete.  $\square$

We continue by bounding the sum on the right of (6.15). Suppose  $t \geq s$ .

$$\begin{aligned} \sum_{m \in \mathbb{Z}} P(A_{1,m}) &= \sum_{m \in \mathbb{Z}} P(X_{[nt]}^0 \geq -[ntb] - [r\sqrt{n}] - m, X_{[ns]}^0 < -[nsb] - [q\sqrt{n}] - m) \\ &= \sum_{m \in \mathbb{Z}} \sum_{\ell > m} P(X_{[ns]}^0 = -[nsb] - [q\sqrt{n}] - \ell) \\ & \quad \times P(X_{[nt]-[ns]}^0 \geq [nsb] - [ntb] + [q\sqrt{n}] - [r\sqrt{n}] - m + \ell) \\ &\stackrel{k=\ell-m}{=} \sum_{m \in \mathbb{Z}} \sum_{k > 0} P(X_{[ns]}^0 = -[nsb] - [q\sqrt{n}] - k - m) \\ & \quad \times P(X_{[nt]-[ns]}^0 \geq [nsb] - [ntb] + [q\sqrt{n}] - [r\sqrt{n}] + k) \\ (6.20) \quad &= \sum_{k > 0} P(X_{[nt]-[ns]}^0 \geq [nsb] - [ntb] + [q\sqrt{n}] - [r\sqrt{n}] + k) \end{aligned}$$

Similarly,

$$(6.21) \quad \sum_{m \in \mathbb{Z}} P(A_{2,m}) = \sum_{k \leq 0} P(X_{[nt]-[ns]}^0 < [nsb] - [ntb] + [q\sqrt{n}] - [r\sqrt{n}] + k).$$

Combining (6.20) and (6.21),

$$\begin{aligned}
(6.22) \quad & \sum_{m \in \mathbb{Z}} [P(A_{1,m}) + P(A_{2,m})] \\
&= \sum_{k > 0} P(X_{[nt]-[ns]}^0 - [nsb] + [ntb] - [q\sqrt{n}] + [r\sqrt{n}] \geq k) \\
&\quad + \sum_{k < 0} P(X_{[nt]-[ns]}^0 - [nsb] + [ntb] - [q\sqrt{n}] + [r\sqrt{n}] \leq k) \\
&= \sum_{k > 0} P(|X_{[nt]-[ns]}^0 - [nsb] + [ntb] - [q\sqrt{n}] + [r\sqrt{n}]| \geq k) \\
&\leq E|X_{[nt]-[ns]}^0 - [nsb] + [ntb]| + |r - q|\sqrt{n} + 1 \\
&\leq C(\sqrt{(t-s)n} + |r - q|\sqrt{n} + 1).
\end{aligned}$$

Combining (6.15) and (6.22) gives this bound for the third term in (6.5):

$$(6.23) \quad \mathbf{E}(|\bar{S}_n(t, r) - \bar{S}_n(s, q)|^{12}) \leq Cn^{-3}[(\sqrt{|t-s|} + |r - q|)\sqrt{n} + 1]^6.$$

Return to (6.5) and apply (6.13), (6.23) and (6.6) to conclude that

$$\begin{aligned}
\mathbf{E}[|\bar{h}_n(t, r) - \bar{h}_n(s, q)|^{12}] &\leq Cn^{-3}[(\sqrt{|t-s|} + |r - q|)\sqrt{n} + 1]^6 \\
&\leq C(|t-s|^\kappa + |r - q|^\kappa + n^{-3}) \leq C(|t-s|^\kappa + |r - q|^\kappa).
\end{aligned}$$

In the last step we used  $t, s, r, q \in [0, 1]$ ,  $2 < \kappa < 3$ , and  $n^{-3} \leq n^{-\gamma\kappa} = \delta_n^\kappa < |(t, r) - (s, q)|^\kappa$ . This completes the proof of Lemma 6.2.  $\square$

The second tightness condition is verified as follows.

LEMMA 6.5. *Under the conditions of Lemma 6.2, for any fixed  $1 < \gamma < 3/2$ , for  $\forall \varepsilon > 0$ ,*

$$\lim_{n \rightarrow \infty} \mathbf{P} \left\{ \sup_{\substack{(t,r),(s,q) \in [0,1]^2 \\ |(t,r)-(s,q)| < n^{-\gamma}}} |\bar{h}_n(t, r) - \bar{h}_n(s, q)| > \varepsilon \right\} = 0.$$

*Proof.* Define the intervals  $I(k) = [(k-1)n^{-\gamma}, (k+1)n^{-\gamma}] \cap [0, 1]$ . For fixed  $\varepsilon > 0$ , first by a union bound and then by decomposition (5.3),

$$\begin{aligned}
(6.24) \quad & \mathbf{P} \left\{ \sup_{\substack{(t,r),(s,q) \in [0,1]^2 \\ |(t,r)-(s,q)| < n^{-\gamma}}} |\bar{h}_n(t, r) - \bar{h}_n(s, q)| > \varepsilon \right\} \\
&\leq \sum_{k_1=1}^{\lfloor n^\gamma \rfloor} \sum_{k_2=1}^{\lfloor n^\gamma \rfloor} \mathbf{P} \left\{ \sup_{t \in I(k_1), r \in I(k_2)} |\mu_0 \bar{H}_n(t, r) - \mu_0 \bar{H}_n(k_1 n^{-\gamma}, k_2 n^{-\gamma})| \geq \frac{\varepsilon}{6} \right\}
\end{aligned}$$

$$(6.25) \quad + \sum_{k_1=1}^{\lfloor n^\gamma \rfloor} \sum_{k_2=1}^{\lfloor n^\gamma \rfloor} \mathbf{P} \left\{ \sup_{t \in I(k_1), r \in I(k_2)} |\bar{S}_n(t, r) - \bar{S}_n(k_1 n^{-\gamma}, k_2 n^{-\gamma})| \geq \frac{\varepsilon}{6} \right\}$$

$$(6.26) \quad + \sum_{k_1=1}^{\lfloor n^\gamma \rfloor} \sum_{k_2=1}^{\lfloor n^\gamma \rfloor} \mathbf{P} \left\{ \sup_{t \in I(k_1), r \in I(k_2)} |\bar{F}_n(t, r) - \bar{F}_n(k_1 n^{-\gamma}, k_2 n^{-\gamma})| \geq \frac{\varepsilon}{6} \right\}.$$

Line (6.24) vanishes for large  $n$  because  $\overline{H}_n(t, r) = O(n^{-1/4})$ .

For the second and third sums (6.25)–(6.26), observe from (5.6) and (5.5) that  $\overline{S}_n(t, r)$  and  $\overline{F}_n(t, r)$  depend on their argument  $(t, r)$  only through  $\lfloor nt \rfloor$ ,  $\lfloor ntb \rfloor$  and  $\lfloor r\sqrt{n} \rfloor$ . For large enough  $n$ , and any  $t \in I(k_1)$ ,  $r \in I(k_2)$ ,

$$\begin{aligned} |nt - k_1 n^{1-\gamma}| &= n|t - k_1 n^{-\gamma}| \leq n^{1-\gamma} < 1/2, \\ |ntb - k_1 n^{1-\gamma} b| &= n|b| \cdot |t - k_1 n^{-\gamma}| \leq |b| n^{1-\gamma} < 1/2, \\ |r\sqrt{n} - k_2 n^{1/2-\gamma}| &= n^{1/2}|r - k_2 n^{-\gamma}| \leq n^{1/2-\gamma} < 1/2. \end{aligned}$$

Thus for  $t \in I(k_1)$  and  $r \in I(k_2)$ , each of  $\lfloor nt \rfloor$ ,  $\lfloor ntb \rfloor$  and  $\lfloor r\sqrt{n} \rfloor$  can have at most one jump. For example,  $\lfloor nt \rfloor$  can only jump from  $\lfloor k_1 n^{1-\gamma} \rfloor - 1$  to  $\lfloor k_1 n^{1-\gamma} \rfloor$  or from  $\lfloor k_1 n^{1-\gamma} \rfloor$  to  $\lfloor k_1 n^{1-\gamma} \rfloor + 1$ . As a result,  $\overline{S}_n(t, r)$  and  $\overline{F}_n(t, r)$  can take only at most 8 different values on  $I(k_1) \times I(k_2)$ .

Suppose  $\{\overline{S}_n(t_i, r_i)\}_{1 \leq i \leq \ell}$  with  $\ell \leq 8$  and  $(t_i, r_i) \in I(k_1) \times I(k_2)$  captures the different values of  $\overline{S}_n(t, r)$  on the square  $I(k_1) \times I(k_2)$ . Then,

$$\begin{aligned} &\mathbf{P} \left\{ \sup_{t \in I(k_1), r \in I(k_2)} |\overline{S}_n(t, r) - \overline{S}_n(k_1 n^{-\gamma}, k_2 n^{-\gamma})| \geq \frac{\varepsilon}{6} \right\} \\ &\leq \sum_{i=1}^{\ell} \mathbf{P} \{ |\overline{S}_n(t_i, r_i) - \overline{S}_n(k_1 n^{-\gamma}, k_2 n^{-\gamma})| \geq \varepsilon/6 \} \\ &\leq C n^{-3} \varepsilon^{-12} \sum_{i=1}^{\ell} \left[ \left( \sqrt{|t_i - k_1 n^{-\gamma}|} + |r_i - k_2 n^{-\gamma}| \right) \sqrt{n} + 1 \right]^6 \\ &\leq C n^{-3} \left[ (n^{-\gamma/2} + n^{-\gamma}) \sqrt{n} + 1 \right]^6 \leq C n^{-3} \end{aligned}$$

where the second inequality comes from Markov's inequality and (6.23), and the third one from  $\gamma > 1$ .

Therefore, for any  $1 < \gamma < 3/2$ , as  $n \rightarrow \infty$ ,

$$\sum_{k_1=1}^{\lfloor n^\gamma \rfloor} \sum_{k_2=1}^{\lfloor n^\gamma \rfloor} \mathbf{P} \left\{ \sup_{\substack{t \in I(k_1) \\ r \in I(k_2)}} |\overline{S}_n(t, r) - \overline{S}_n(k_1 n^{-\gamma}, k_2 n^{-\gamma})| \geq \frac{\varepsilon}{6} \right\} \leq C n^{2\gamma-3} \rightarrow 0.$$

Same reasoning with inequality (6.13) shows that line (6.26) vanishes as  $n \rightarrow \infty$ . Lemma 6.5 has been proved.  $\square$

The proof of Theorem 3.4 is complete.

## APPENDIX A. AN ERGODIC LEMMA

In this section  $p(x, y) = p(0, y - x)$  is an arbitrary random walk kernel on  $\mathbb{Z}^d$  with the property that the smallest subgroup that contains the support  $\{x : p(0, x) > 0\}$  is  $\mathbb{Z}^d$  itself.

LEMMA A.1. *Transition probability  $p(x, y)$  has no bounded harmonic functions other than constants.*



*Proof.* Suppose  $h$  is a bounded harmonic function on  $\mathbb{Z}^d$ , that is,  $h(x) = \sum_y p(x, y)h(y)$  for all  $x \in \mathbb{Z}^d$ . Suppose  $p(u, v) > 0$ . Then the coupling described on p. 69 of [23] works to show that  $h(u) = h(v)$ . (This coupling is also spelled out in Section 1.5 of [31].) By assumption for any  $x, y \in \mathbb{Z}^d$  there exists a path  $x = x_0, x_1, \dots, x_m = y$  in  $\mathbb{Z}^d$  such that  $p(x_i, x_{i+1}) + p(x_{i+1}, x_i) > 0$  for each  $i = 0, \dots, m-1$ , and thereby  $h(x) = h(x_0) = \dots = h(x_m) = h(y)$ .  $\square$

Let  $\mathcal{S}$  be a Polish space,  $\Gamma = \mathcal{S}^{\mathbb{Z}^d}$ , and shifts  $(\theta_x \zeta)(y) = \zeta(x + y)$  for  $\zeta \in \Gamma$  and  $x, y \in \mathbb{Z}^d$ .

LEMMA A.2. *Let  $\nu$  be a probability measure on  $\Gamma$  that is invariant and ergodic under the shift group and  $f \in L^1(\nu)$  with finite mean  $E^\nu[f]$ . For  $x \in \mathbb{Z}^d$ ,  $t \in \mathbb{Z}_+$ , and  $\zeta \in \Gamma$ , define*

$$(A.1) \quad g_t(x, \zeta) = \sum_{y \in \mathbb{Z}} p^t(x, y) f(\theta_y \zeta).$$

*Then  $\forall x \in \mathbb{Z}^d$ ,  $g_t(x, \zeta) \rightarrow E^\nu[f]$  in  $L^1(\nu)$  as  $t \rightarrow \infty$ .*

*Proof.* The argument is a Fourier analytic one suggested by the proof on p. 30-31 of [24]. The characteristic function of the jump probability is

$$\phi_X(\alpha) = \sum_{y \in \mathbb{Z}} p(0, y) e^{i\alpha \cdot y}, \quad \alpha \in \mathbb{R}^d.$$

For  $0 < r < \infty$  define truncated functions  $f_r(\zeta) = (-r) \vee (f(\zeta) \wedge r)$  and  $g_t(x, r, \zeta) = \sum_{y \in \mathbb{Z}} p^t(x, y) f_r(\theta_y \zeta)$ . Note that  $g_t(x, r, \zeta)$  is bounded, uniformly over  $x$  and  $\zeta$ . We show that, as  $t \rightarrow \infty$ ,  $g_t(x, r, \zeta)$  converges in  $L^2(\nu)$  to a constant. An  $L^1$  approximation via the truncation then implies the result.

The function  $V^{(r)}(x) = E^\nu[f_r(\zeta) f_r(\theta_x \zeta)]$  is nonnegative definite (i.e.  $\sum_{x, y} V^{(r)}(x - y) z_x \overline{z_y} \geq 0$  for any choice of finitely many complex numbers  $\{z_x\}$ ). By Herglotz' Theorem (Chapter XIX.6 in Feller [11]), there exists a bounded measure  $\gamma$  on  $[-\pi, \pi)^d$  such that

$$V^{(r)}(x) = \int e^{-i x \cdot \alpha} \gamma(d\alpha), \quad x \in \mathbb{Z}^d.$$

Let  $X_t$  and  $\tilde{X}_t$  be i.i.d. copies of the random walk with transition  $p$ . Compute:

$$\begin{aligned} \int g_t(x, r, \zeta) g_s(x, r, \zeta) \nu(d\zeta) &= \int E^x[f_r(\theta_{X_t} \zeta)] E^x[f_r(\theta_{\tilde{X}_s} \zeta)] \nu(d\zeta) \\ &= E^{(x, x)} \int f_r(\theta_{X_t} \zeta) f_r(\theta_{\tilde{X}_s} \zeta) \nu(d\zeta) = E^{(x, x)} [V^{(r)}(\tilde{X}_s - X_t)] \\ &= E^{(x, x)} \int e^{-i\alpha \cdot (\tilde{X}_s - X_t)} \gamma(d\alpha) = \int \overline{E^x e^{i\alpha \cdot \tilde{X}_s}} \cdot E^x e^{i\alpha \cdot X_t} \gamma(d\alpha) \\ &= \int [\overline{\phi_X(\alpha)}]^s [\phi_X(\alpha)]^t \gamma(d\alpha) = \int [\phi_X(\alpha)]^s [\overline{\phi_X(\alpha)}]^t \gamma(d\alpha). \end{aligned}$$

The last equality came by switching  $s$  and  $t$  around in the calculation. From this,

$$\begin{aligned}
& \int [g_t(x, r, \zeta) - g_s(x, r, \zeta)]^2 \nu(d\zeta) \\
&= \int [g_t(x, r, \zeta)^2 - 2g_t(x, r, \zeta)g_s(x, r, \zeta) + g_s(x, r, \zeta)^2] \nu(d\zeta) \\
&= \int [|\phi_X(\alpha)|^{2t} - 2\overline{\phi_X(\alpha)^s} \phi_X(\alpha)^t + |\phi_X(\alpha)|^{2s}] \gamma(d\alpha) = \int |\phi_X(\alpha)^t - \phi_X(\alpha)^s|^2 \gamma(d\alpha) \\
\text{(A.2)} \quad &= \int_{\alpha \neq 0} |\phi_X(\alpha)^t - \phi_X(\alpha)^s|^2 \gamma(d\alpha).
\end{aligned}$$

Since  $\mathbb{Z}^d$  is the smallest subgroup that contains the support of  $p$ ,  $|\phi_X(\alpha)| < 1 \forall \alpha \in [-\pi, \pi)^d \setminus \{0\}$  (T7.1 in [34]). Thus the integrand in (A.2) is bounded and converges to zero as  $s, t \rightarrow \infty$ . Hence  $\{g_t(x, r, \zeta)\}_{t \in \mathbb{Z}_+}$  is Cauchy in  $L^2(\nu)$  and  $\exists$  a bounded  $L^2(\nu)$  limit  $\bar{g}(x, r, \zeta) = \lim_{t \rightarrow \infty} g_t(x, r, \zeta)$ .

Letting  $s \rightarrow \infty$  in  $g_{s+t}(x, r, \zeta) = \sum_y p^t(x, y) g_s(y, r, \zeta)$  implies

$$\bar{g}(x, r, \zeta) = \sum_y p^t(x, y) \bar{g}(y, r, \zeta).$$

Hence for almost every  $\zeta$ ,  $\bar{g}(\cdot, r, \zeta)$  is a bounded harmonic function for  $p(x, y)$  and thereby a constant in  $x$ . Combining this with a shift gives  $\bar{g}(0, r, \zeta) = \bar{g}(x, r, \zeta) = \bar{g}(0, r, \theta_x \zeta) \forall x \in \mathbb{Z}^d$ . By ergodicity,  $\bar{g}(0, r, \zeta)$  equals  $\nu$ -almost surely a constant, and hence  $\bar{g}(x, r, \zeta)$  equals  $\nu$ -almost surely the same constant for all  $x \in \mathbb{Z}^d$ .

Now we transfer these properties to  $g_t(x, \zeta)$  through  $L^1$  approximation. By shift-invariance

$$\|g_t(x, \zeta) - g_t(x, r, \zeta)\|_1 \leq \sum_y p^t(x, y) \|f \circ \theta_y - f_r \circ \theta_y\|_1 = \|f - f_r\|_1 \xrightarrow{r \rightarrow \infty} 0.$$

From this we deduce that  $\{g_t(x, \cdot)\}$  is a Cauchy sequence in  $L^1(\nu)$ . Hence we have an  $L^1(\nu)$  limit  $g(x, \zeta) = \lim_{t \rightarrow \infty} g_t(x, \zeta)$  and we can take  $t \rightarrow \infty$  in the bound above to get

$$\|g(x, \zeta) - \bar{g}(x, r, \zeta)\|_1 \leq \|f - f_r\|_1.$$

Letting  $r \rightarrow \infty$  takes the right-hand side to zero, and we conclude that  $g(x, \zeta)$  is  $\nu$ -almost surely a constant independent of  $x$ . This constant must equal  $E^\nu[f]$  because by the  $L^1(\nu)$  convergence  $E^\nu[g(x, \cdot)] = \lim_{t \rightarrow \infty} E^\nu[g_t(x, \cdot)] = E^\nu[f]$ .  $\square$

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